PROPOSED SOLUTIONS
HOMEWORK 7
GRA6039 ECONOMETRICS WITH PROGRAMMING
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Solutions to Ex. 1. **(a)** The two hypotheses about $p = \Pr(\text{coin showing heads})$ are

$$H_0\colon p = \frac{1}{2}, \quad \text{vs.} \quad H_A\colon p \neq \frac{1}{2}.$$

(b) The probability og getting the sequence in (**??**) is the same as getting any other sequence, namely

$$\frac{1}{2^6} = \frac{1}{64} = 0.015625.$$

The fact that this probability is small is not evidence against H_0. The reason being that any sequence of six tosses has this probability. For example, the probability under H_0 of getting H, T, H, T, H, T – which definitely looks like something that could have been produced by a fair coin, is also $1/2^6$. **(c)** The number of different sequences you can get in six tosses is 64. Make a drawing like the following and connect the outcomes in in the first row to the two below, and so on (this corresponds to three tosses),

$$\begin{array}{cccccccc} & & & T & H & & & \\ & & H & T & H & T & & \\ H & T & H & T & H & T & H & T \end{array}$$

and you quickly realise that the number of possible sequences is

$$2^{\text{number of tosses}}.$$

There are therefore $2^6 = 64$ different sequences of heads and tails that you can get i six tosses. **(d)** There are two types of sequence that are *just as* surprising as the the one in (**??**): All the sequences that contain one tail and seven heads:

$$T\,H\,H\,H\,H\,H$$
$$H\,T\,H\,H\,H\,H$$
$$H\,H\,T\,H\,H\,H$$
$$H\,H\,H\,T\,H\,H$$
$$H\,H\,H\,H\,T\,H$$
$$H\,H\,H\,H\,H\,T$$

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As you can see, there are six of these. But under H_0, you would be just as surprised by

$$HTTTTT$$
$$THTTTT$$
$$TTHTTT$$
$$TTTHTT$$
$$TTTTHT$$
$$TTTTTH$$

and you would be even *more surprised* by

$$HHHHHH \quad \text{and} \quad TTTTTT$$

Add this up and get 14. **(e)** The probability is getting a sequence that is *just as* or *even more* surprising than the sequence in (**??**) is therefore

$$\frac{14}{64} = \frac{7}{32} = 0.21875.$$

This probability is what is commonly called a p-value, and if you are testing your H_0 with a 0.05 threshold, you do not reject H_0.

Solutions to Ex. 2. **(a)** Prove Lemma 1 from the lecture: If $X \sim \mathrm{N}(a, b^2)$, then

$$\frac{X-a}{b} \sim \mathrm{N}(0,1).$$

Since $X \sim \mathrm{N}(a, b^2)$, we know that its cdf is

$$\Pr(X \leq x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}b} \exp\{-\frac{1}{2}(\frac{y-a}{b})^2\}\,\mathrm{d}y.$$

To find the distribution of $(X-a)/b$, we find its cdf.

$$\Pr(\frac{X-a}{b} \leq z) = \Pr(X \leq bz + a) = \int_{-\infty}^{bz+a} \frac{1}{\sqrt{2\pi}b} \exp\{-\frac{1}{2}(\frac{y-a}{b})^2\}\,\mathrm{d}y$$
$$= \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp(-w^2/2)\,\mathrm{d}w = \Phi(z),$$

substituting $w = (y-a)/b$, so that $\mathrm{d}y = b\,\mathrm{d}w$, and $w \to -\infty$ when $y \to -\infty$, and $w = z$ when $y = bz + a$.

(b) If $X \sim \mathrm{N}(a, b^2)$, then

$$\Pr(X \leq x) = \Pr\big(\frac{X-a}{b} \leq \frac{x-a}{b}\big) = \Phi\big(\frac{x-a}{b}\big),$$

using the result from (a).

(c) Let $X_1, \ldots, X_n$ be i.i.d. random variables with the $\mathrm{N}(\mu, \sigma^2)$ distribution. We know from Lemma 7.2 in the lecture notes that a linear combination of independent normally distributed random variables, is a normally distributed random variable. The empirical average $\bar{X}_n = (1/n)\sum_{i=1}^{n} X_i$ is a linear combination of the $X_1, \ldots, X_n$, and therefore has a normal distribution. Its expectation is $\mathrm{E}\,\bar{X}_n = \mu$, and $\mathrm{Var}(\bar{X}_n) = \sigma^2/n$, and we conclude that

$$\bar{X}_n \sim \mathrm{N}(\mu, \sigma^2/n).$$

It also follows from Lemma 7.2 that

$$\bar{X}_n - \mu \sim \mathrm{N}(0, \sigma^2/n).$$

In that lemma, set all the $\gamma_i = 0$, except one of them, and you'll see it. Finally, that

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim \mathrm{N}(0, 1),$$

follows from (a), by setting $a = \mu$ and $b^2 = \sigma^2/n$.
(d) Since $\bar{X}_n \sim \mathrm{N}(\mu, \sigma^2/n)$, the claim is just a reformulation of (b). Set $a = \mu$ and $b^2 = \sigma^2/n$.
(e) The matlab code is

```
z = 1.645;
normcdf(-z,0,1)
1 - normcdf(z,0,1)
norminv(normcdf(-z,0,1),0,1)
norminv(normcdf(z,0,1),0,1)
```

(f) A 90 percent confidence interval for μ is

$$[\bar{X}_n - \Phi^{-1}(0.95)\frac{\sigma}{\sqrt{n}}, \bar{X}_n + \Phi^{-1}(0.95)\frac{\sigma}{\sqrt{n}}].$$

Solutions to Ex. 3.

$$H_0\colon \theta \geq \theta_{\mathrm{nox}}, \quad \text{vs.} \quad H_A\colon \theta < \theta_{\mathrm{nox}}$$

(a) We can only control one of the error-probabilities, and we construct our hypotheses so that we control the probability of rejecting a true null-hypothesis. We see it as more serious to tell people to swim when the water is noxious, than telling them to stay at home when the water is fine. Therefore, the hypotheses are formulated as above.

(b) A test for the null-hypothesis is: Reject H_0 if

$$\bar{X}_n - \theta_{\mathrm{nox}} \leq c_n,$$

for some number c_n. The c_n ensuring that the probability of rejecting H_0 when $\mathrm{E}\,[X_1] = \theta = \theta_{\mathrm{nox}}$ is 0.025, is

$$c_n = \frac{\sigma\Phi^{-1}(0.025)}{\sqrt{n}} = -\frac{\sigma 1.96}{\sqrt{n}}.$$

To see this, we do our computations under the hypothesis that $\theta = \theta_{\mathrm{nox}}$, then

$$\begin{aligned}\Pr_{\theta_{\mathrm{nox}}}(\bar{X}_n - \theta_{\mathrm{nox}} \leq c_n) &= \Pr_{\theta_{\mathrm{nox}}}\big(\bar{X}_n - \theta_{\mathrm{nox}} \leq \frac{\sigma\Phi^{-1}(0.025)}{\sqrt{n}}\big) \\ &= \Pr_{\theta_{\mathrm{nox}}}\big(\sqrt{n}(\bar{X}_n - \theta_{\mathrm{nox}})/\sigma \leq \Phi^{-1}(0.025)\big) \\ &= \Phi(\Phi^{-1}(0.025)) = 0.025,\end{aligned}$$

where we use that $\sqrt{n}(\bar{X}_n - \theta_{\rm nox})/\sigma \sim {\rm N}(0,1)$ when $\theta = \theta_{\rm nox}$. As we are soon to see, ${\rm Pr}_\theta(\bar{X}_n - \theta_{\rm nox} \leq c_n) \leq 0.025$ whenever $\theta \geq \theta_{\rm nox}$.

(c) Find an expression for the power function

$$
\begin{aligned}
{\rm power}(\theta) &= {\rm Pr}_\theta(\bar{X}_n - \theta_{\rm nox} \leq c_n) = {\rm Pr}_\theta(\bar{X}_n - \theta + \theta - \theta_{\rm nox} \leq c_n) \\
&= {\rm Pr}_\theta(\bar{X}_n - \theta \leq c_n - (\theta - \theta_{\rm nox})) \\
&= {\rm Pr}_\theta(\sqrt{n}(\bar{X}_n - \theta)/\sigma \leq \sqrt{n}c_n/\sigma - \sqrt{n}(\theta - \theta_{\rm nox})/\sigma) \\
&= \Phi\big(\Phi^{-1}(0.025) - \sqrt{n}(\theta - \theta_{\rm nox})/\sigma\big).
\end{aligned}
$$

where we use that $\sqrt{n}(\bar{X}_n - \theta)/\sigma \sim {\rm N}(0,1)$ when ${\rm E}\, X_1 = \theta$.

(d) Here is Matlab code used to graph the power function.

```
n = 10;
theta_nox = 0.234;
sigma2 = 1.23;
theta = linspace(theta_nox - 2,theta_nox + 2,500);
alpha = 0.025

power = normcdf( norminv(alpha) - sqrt(n)*(theta - theta_nox)/sqrt(sigma2));

plot(theta,power,"LineWidth",2)
ylabel("Power");xlabel("theta");
hold on
plot([min(theta),max(theta)],[alpha,alpha])
plot([theta_nox,theta_nox],[0,1])
```

Solutions to Ex. 4. We have i.i.d. random variables $X_1, \ldots, X_n$ with known variance 1, but do not know what their expectation ${\rm E}\,[X_1] = \theta$ is. Test the hypothesis that

$$H_0\colon \theta = 0,$$

versus the alternative $H_A\colon \theta \neq 0$.

(a) Under H_0, that is, when $\theta = 0$, $\sqrt{n}\bar{X}_n \sim {\rm N}(0,1)$. With

$$c_n = \frac{\Phi^{-1}(0.95)}{\sqrt{n}}.$$

we compute the probability of rejecting H_0 when H_0 is true.

$$
\begin{aligned}
{\rm Pr}_0(\text{reject } H_0) &= {\rm Pr}_0(\bar{X}_n \leq -c_n \text{ or } \bar{X}_n \geq c_n) = {\rm Pr}_0(\bar{X}_n \leq -c_n) + \{1 - {\rm Pr}_0(\bar{X}_n \leq c_n)\} \\
&= {\rm Pr}_0(\sqrt{n}\bar{X}_n \leq -\sqrt{n}c_n) + \{1 - {\rm Pr}_0(\sqrt{n}\bar{X}_n \leq \sqrt{n}c_n)\} \\
&= \Phi(-\sqrt{n}c_n) + \{1 - \Phi(\sqrt{n}c_n)\} = 2\Phi(-\sqrt{n}c_n) \\
&= 2\Phi(-\Phi^{-1}(0.95)) = 2\Phi(\Phi^{-1}(0.05)) = 2 \times 0.05 = 0.10,
\end{aligned}
$$

where we use the symmetry of the normal distribution several times.

(b) The power function is

$$\begin{aligned}\Pr_\theta(\text{reject } H_0) &= \Pr_\theta(\bar{X}_n \le -c_n \text{ or } \bar{X}_n \ge c_n) = \Pr_\theta(\bar{X}_n \le -c_n) + \{1 - \Pr_\theta(\bar{X}_n \le c_n)\} \\ &= \Pr_\theta(\sqrt{n}(\bar{X}_n - \theta) \le -\sqrt{n}(c_n + \theta)) + \{1 - \Pr_\theta(\sqrt{n}(\bar{X}_n - \theta) \le \sqrt{n}(c_n - \theta))\} \\ &= \Phi(-\sqrt{n}(c_n + \theta)) + 1 - \Phi(\sqrt{n}(c_n - \theta)) \\ &= \Phi(-\Phi^{-1}(0.95) - \sqrt{n}\theta)) + 1 - \Phi(\Phi^{-1}(0.95) - \sqrt{n}\theta).\end{aligned}$$

(c) Here is Matlab code to plot the power function for $n = 10$ and for $n = 33$.

```
theta = linspace(-2,2,500);
% I split it up to make it easier to read
pwr10_1 = normcdf(-norminv(0.95) - sqrt(10)*theta);
pwr10_2 = 1 - normcdf(norminv(0.95) - sqrt(10)*theta);
power10 = pwr10_1 + pwr10_2;

pwr33_1 = normcdf(-norminv(0.95) - sqrt(33)*theta);
pwr33_2 = 1 - normcdf(norminv(0.95) - sqrt(33)*theta);
power33 = pwr33_1 + pwr33_2;

plot(theta,power10,"LineWidth",2)
ylim([0,1]);ylabel("Power");xlabel("theta");
hold on
plot(theta,power33,"LineWidth",2)
plot([min(theta),max(theta)],[0.1,0.1])
```

(d) Compute the two probabilities in Matlab

```
theta = 1/2

pwr10_1 = normcdf(-norminv(0.95) - sqrt(10)*theta);
pwr10_2 = 1 - normcdf(norminv(0.95) - sqrt(10)*theta);
power10 = pwr10_1 + pwr10_2 % 0.4752

pwr33_1 = normcdf(-norminv(0.95) - sqrt(33)*theta);
pwr33_2 = 1 - normcdf(norminv(0.95) - sqrt(33)*theta);
power33 = pwr33_1 + pwr33_2 % 0.8902
```

Which means that $\Pr_{1/2}(\text{reject } H_0) = 0.4752$ when $n = 10$, and it is $\Pr_{1/2}(\text{reject } H_0) = 0.8902$ when $n = 33$. An increasing sample size makes our test more powerful: The probability that we detect that H_0 is false increases.

(e)

```
x = [-0.1887 -0.3978 2.7470 0.4135 0.1691 1.6996 1.2608 0.1342 -0.1759 0.4977];
n = length(x);
cn = norminv(0.95)/sqrt(n);
(mean(x) <= -cn)|(mean(x) >= cn) % is True
```

Since $\bar{X}_n = 0.6159 \geq \Phi^{-1}(0.95)/\sqrt{10} = 0.5201$, we reject the null-hypothesis at the 10 percent significance level.

Solutions to Ex. 5. (a)

```
n = 10^4;
tt = linspace(1/n,1,n);
Delta = 1/n;

xi = normrnd(0,1,1,n);

Zt = sqrt(Delta).*cumsum(xi);

% Experiment with this
mu = 0.123;
sigma = sqrt(0.02);
S0 = 17;

St = S0.*exp(mu.*tt + sigma.*Zt);

plot(tt,St,"Linewidth",1.41)
```

(b)

```
rr = linspace(-0.99,0.99,10^3)

plot(rr,rr,"Linewidth",1.41)
hold on
plot(rr,log(rr + 1),"Color","r","Linewidth",1.41)
```

From the plot made in this matlab-script we see that $rr \approx \log(1 + rr)$ when rr is close to zero. Therefore, when the returns $R(t_j) = (S_{t_j} - S_{t_{j-1}})/S_{t_{j-1}}$ (perhaps because the intervals $[t_{j-1}, t_j]$ are small or the volatility is not too large)

$$\begin{aligned} Y_{t_j} - Y_{t_{j-1}} &= \log S_{t_j} - \log S_{t_{j-1}} = \log \frac{S_{t_j}}{S_{t_{j-1}}} = \log \frac{S_{t_j} - S_{t_{j-1}} + S_{t_{j-1}}}{S_{t_{j-1}}} \\ &= \log \big(\frac{S_{t_j} - S_{t_{j-1}}}{S_{t_{j-1}}} + 1\big) = \log(R(t_j) + 1) \approx R(t_j). \end{aligned}$$

(c) We see that

$$Y_{t_j} - Y_{t_{j-1}} = \mu\Delta_n + \sigma\Delta_n^{1/2}\xi_{t_j}$$

where $\xi_{t_j} \sim \mathrm{N}(0, 1)$, so by Lemma 7.2 in the Lecture notes $Y_{t_j} - Y_{t_{j-1}}$ is normally distributed. Its expectation and variance

$$\begin{aligned} &\mathrm{E}\,[Y_{t_j} - Y_{t_{j-1}}] = \mu\Delta_n + \sigma\Delta_n^{1/2}\mathrm{E}\,[\xi_{t_j}] = \mu\Delta_n, \\ &\mathrm{Var}(Y_{t_j} - Y_{t_{j-1}}) = \mathrm{Var}(\sigma\Delta_n^{1/2}\xi_{t_j}) = \sigma^2\Delta_n\mathrm{Var}(\xi_{t_j}) = \sigma^2\Delta_n. \end{aligned}$$

(d) The estimator $\widehat{\mu}_n$ is

$$\widehat{\mu}_n = \frac{1}{n\Delta_n}\sum_{j=1}^{n}(Y_{t_j} - Y_{t_{j-1}}) = \sum_{j=1}^{n}(Y_{t_j} - Y_{t_{j-1}}) = Y_{t_n} - Y_{t_0} = \mu t_n + \sigma Z_{t_n} = \mu + \sigma Z_{t_n},$$

since $n\Delta_n = n/n = 1$, and $t_n = 1$. The expectation of Z_{t_n} is zero, so $\mathrm{E}\,\widehat{\mu}_n = \mu$, it is unbiased. But

$$\mathrm{Var}(\widehat{\mu}_n) = \sigma^2 \mathrm{Var}(Z_{t_n}) = \sigma^2 \mathrm{Var}(\Delta_n^{1/2} \sum_{i=1}^n \xi_{t_i}) = \sigma^2 \Delta_n n = \sigma^2,$$

is the same for all n, so $\widehat{\mu}_n$ cannot be consistent.

(e) We are to show that the *realised volatilty*,

$$\widehat{\sigma}_n^2 = \sum_{j=1}^n (Y_{t_j} - Y_{t_{j-1}})^2.$$

is consistent for the volatility σ^2. By inserting $Y_{t_j} - Y_{t_{j-1}} = \Delta_n \mu + \sigma_s \Delta^{1/2} \xi_{t_j}$ and expanding the square, we find

$$\widehat{\sigma}_n^2 = \mu^2 \Delta_n + 2\mu\sigma\Delta_n^{3/2} \sum_{j=1}^n \xi_{t_j} + \sigma^2 \Delta_n \sum_{j=1}^n \xi_{t_j}^2,$$

Let's look at this term by term. Recall that $\Delta_n = 1/n$, so $\mu^2 \Delta_n \to 0$ as $n \to \infty$. For the second term

$$2\mu\sigma\Delta_n^{3/2} \sum_{j=1}^n \xi_{t_j} = 2\mu\sigma \frac{1}{\sqrt{n}} \frac{1}{n} \sum_{j=1}^n \xi_{t_j} \overset{p}{\to} 0,$$

by the Law of large numbers, because the $\xi_{t_1}, \dots, \xi_{t_n}$ are i.i.d. $\mathrm{N}(0,1)$, so $(1/n)\sum_{j=1}^n \xi_{t_j} \to_p \mathrm{E}\,[\xi_{t_1}] = 0$ by the Law of large numbers, and $1/\sqrt{n} \to 0$. The claim then follows from PLIM.2. The last term is

$$\sigma^2 \Delta_n \sum_{j=1}^n \xi_{t_j}^2 = \sigma^2 \frac{1}{n} \sum_{j=1}^n \xi_{t_j}^2 \overset{p}{\to} \sigma^2,$$

because $\xi_{t_1}^2, \dots, \xi_{t_n}^2$ are i.i.d. with expectation 1 and variance 2. Then $(1/n)\sum_{j=1}^n \xi_{t_j}^2 \to_p \mathrm{E}\,[\xi_{t_1}^2] = 1$ by the LLN. We can now use Property PLIM.2 (Wooldridge (2019, p. 723) and Lemma 5.2 in the Lecture notes, in particular Remark 5.3), and conclude that $\widehat{\sigma}_n^2 \to_p \sigma^2$.

Solutions to Ex. 6. This exercise builds on the previous one.

(a) We have that ξ_{t_j} is a standard normal, and that

$$\eta_{t_j} = \rho\xi_{t_j} + (1-\rho^2)^{1/2}\epsilon_{t_j},$$

with ϵ_{t_j} a standard normal independent of ξ_{t_j}. Then

$$\begin{aligned}
\mathrm{Cov}(\xi_{t_j}, \eta_{t_j}) &= \mathrm{E}\,[\xi_{t_j}\eta_{t_j}] - \mathrm{E}\,[\xi_{t_j}]\,\mathrm{E}\,[\eta_{t_j}] = \mathrm{E}\,[\xi_{t_j}\eta_{t_j}] \\
&= \mathrm{E}\,[\xi_{t_j}(\rho\xi_{t_j} + (1-\rho^2)^{1/2}\epsilon_{t_j})] = \rho\,\mathrm{E}\,[\xi_{t_j}^2] + (1-\rho^2)^{1/2}\mathrm{E}\,[\xi_{t_j}\epsilon_{t_j}] \\
&= \rho\,\mathrm{E}\,[\xi_{t_j}^2] = \rho,
\end{aligned}$$

because $\mathrm{E}\,[\xi_{t_j}\epsilon_{t_j}] = \mathrm{E}\,[\xi_{t_j}]\,\mathrm{E}\,[\epsilon_{t_j}] = 0$ by independence, and $\mathrm{E}\,[\xi_{t_j}^2] = \mathrm{Var}(\xi_{t_j}) = 1$.

(b) $Z_{t_j} - Z_{t_{j-1}} = \Delta_n^{1/2}\sum_{i=1}^{j}\xi_{t_i} - \Delta_n^{1/2}\sum_{i=1}^{j-1}\xi_{t_i} = \Delta_n^{1/2}\xi_{t_j}$. Since $\xi_{t_j} \sim \mathrm{N}(0,1)$, the random variable $\Delta_n^{1/2}\xi_{t_j}$ must also have a normal distribution (by Lemma 7.2 in the

Lecture notes). Its expectation is $\mathrm{E}\,[\Delta_n^{1/2}\xi_{t_j}] = \Delta_n^{1/2}\mathrm{E}\,[\xi_{t_j}] = 0$, and its variance is $\mathrm{Var}(\Delta_n^{1/2}\xi_{t_j}) = \Delta_n \mathrm{Var}(\xi_{t_j}) = \Delta_n$. Therefore $\Delta_n^{1/2}\xi_{t_j} \sim \mathrm{N}(0, \Delta_n)$. Since $\xi_{t_1}, \ldots, \xi_{t_n}$ are independent, so are $\Delta_n^{1/2}\xi_{t_1}, \ldots, \Delta_n^{1/2}\xi_{t_n}$. The same argument applies to the $W_{t_j} - W_{t_{j-1}}$.

(c) We have that $Z_{t_j} - Z_{t_{j-1}} = \Delta_n^{1/2}\xi_{t_j}$ and that $W_{t_j} - W_{t_{j-1}} = \Delta_n^{1/2}\eta_{t_j}$. Then

$$\mathrm{Cov}(Z_{t_j} - Z_{t_{j-1}}, W_{t_j} - W_{t_{j-1}}) = \mathrm{Cov}(\Delta_n^{1/2}\xi_{t_j}, \Delta_n^{1/2}\eta_{t_j}) = \Delta_n \mathrm{E}\,[\xi_{t_j}\eta_{t_j}] = \Delta_n \rho,$$

where we use the result from (a). Moreover, $Y_{t_j} - Y_{t_{j-1}} = \Delta_n\mu_S + \sigma_S\Delta_n^{1/2}\xi_{t_j}$, and $X_{t_j} - X_{t_{j-1}} = \Delta_n\mu_C + \sigma_C\Delta_n^{1/2}\eta_{t_j}$. Then $\mathrm{E}\,[Y_{t_j} - Y_{t_{j-1}}] = \Delta_n\mu_S$, and $\mathrm{E}\,[X_{t_j} - X_{t_{j-1}}] = \Delta_n\mu_C$, so

$$\begin{aligned}\mathrm{Cov}(Y_{t_j} - Y_{t_{j-1}}, X_{t_j} - X_{t_{j-1}}) &= \mathrm{E}\,(Y_{t_j} - Y_{t_{j-1}} - \Delta_n\mu_S)(X_{t_j} - X_{t_{j-1}} - \Delta_n\mu_C) \\ &= \mathrm{E}\,[\sigma_S\Delta_n^{1/2}\xi_{t_j}\sigma_C\Delta_n^{1/2}\eta_{t_j}] = \Delta_n\sigma_S\sigma_C\,\mathrm{E}\,[\xi_{t_j}\eta_{t_j}] = \Delta_n\rho\sigma_S\sigma_C,\end{aligned}$$

using the result from (a).
(d) An estimator for $\sigma_S\sigma_C\rho$ is

$$\widehat{\mathrm{cov}}_n = \sum_{j=1}^{n}(X_{t_j} - X_{t_{j-1}})(Y_{t_j} - Y_{t_{j-1}}).$$

Write

$$\widehat{\mathrm{cov}}_n = \mu_S\mu_C\Delta_n + \mu_S\Delta_n^{3/2}\sum_{j=1}^{n}\eta_{t_j} + \mu_C\Delta_n^{3/2}\sum_{j=1}^{n}\xi_{t_j} + \sigma_S\sigma_C\Delta_n\sum_{j=1}^{n}\xi_{t_j}\eta_{t_j}.$$

To show that $\widehat{\mathrm{cov}}_n \to_p \rho\sigma_S\sigma_C$, we look at this expression term by term, and use PLIM.2. The first term $\mu_S\mu_C\Delta_n = \mu_S\mu_C/n \to 0$. The second term

$$\mu_S\Delta_n^{3/2}\sum_{j=1}^{n}\eta_{t_j} = \mu_S\frac{1}{\sqrt{n}}\frac{1}{n}\sum_{j=1}^{n}\eta_{t_j} \overset{p}{\to} 0,$$

by the LLN, because $\eta_{t_1}, \ldots, \eta_{t_n}$ are i.i.d. standard normals, so $(1/n)\sum_{j=1}^{n}\eta_{t_j} \to_p \mathrm{E}\,\eta_{t_1} = 0$, and similarly $\mu_C\Delta_n^{3/2}\sum_{j=1}^{n}\xi_{t_j} \to_p 0$. For the final term

$$\sigma_S\sigma_C\Delta_n\sum_{j=1}^{n}\xi_{t_j}\eta_{t_j}\sigma_S\sigma_C\frac{1}{n}\sum_{j=1}^{n}\xi_{t_j}\eta_{t_j} \overset{p}{\to} \rho\sigma_S\sigma_C.$$

This is because $\xi_{t_1}\eta_{t_1}, \ldots, \xi_{t_n}\eta_{t_n}$ are i.i.d. random variables with mean ρ and finite variance. The LLN therefore gives that $(1/n)\sum_{j=1}^{n}\xi_{t_j}\eta_{t_j} \to_p \mathrm{E}\,[\xi_{t_1}\eta_{t_1}] = \rho$. It now follows from PLIM.2 that $\widehat{\mathrm{cov}}_n \to_p \rho\sigma_S\sigma_C$.
(e) This is about finding the least squares estimator. Differentiate $g(\beta)$, set the derivative equal to zero, and solve for β.
(f) We have the estimator

$$\widehat{\beta}_n = \frac{\sum_{j=1}^{n}(X_{t_j} - X_{t_{j-1}})(Y_{t_j} - Y_{t_{j-1}})}{\sum_{j=1}^{n}(X_{t_j} - X_{t_{j-1}})^2} = \frac{\widehat{\mathrm{cov}}_n}{\widehat{\sigma}_{C,n}^2},$$

where $\widehat{\sigma}^2_{C,n} = \sum_{j=1}^n (X_{t_j} - X_{t_{j-1}})^2$. From Ex. 5 we know that $\widehat{\sigma}^2_{C,n} \to_p \sigma^2_C$, and from (d) that $\widehat{\mathrm{cov}}_n \to_p \rho\sigma_S\sigma_C$. Property PLIM.2 then yields

$$\widehat{\beta}_n = \frac{\widehat{\mathrm{cov}}_n}{\widehat{\sigma}^2_{C,n}} \overset{p}{\to} \frac{\rho\sigma_C\sigma_S}{\sigma^2_C} = \frac{\rho\sigma_S}{\sigma_C}.$$

(g) A consistent estimator for ρ is

$$\widehat{\rho}_n = \Big(\frac{\sum_{j=1}^n (X_{t_j} - X_{t_{j-1}})^2}{\sum_{j=1}^n (Y_{t_j} - Y_{t_{j-1}})^2}\Big)^{1/2} \widehat{\beta}_n.$$

Since $\sum_{j=1}^n (X_{t_j} - X_{t_{j-1}})^2 \to_p \sigma^2_C$ and $\sum_{j=1}^n (Y_{t_j} - Y_{t_{j-1}})^2 \to_p \sigma^2_S$, their ratio tends in probability to σ^2_C/σ^2_S, using PLIM2. Since $g(x) = x^{1/2}$ is a continuous function, PLIM.1 then gives that

$$\Big(\frac{\sum_{j=1}^n (X_{t_j} - X_{t_{j-1}})^2}{\sum_{j=1}^n (Y_{t_j} - Y_{t_{j-1}})^2}\Big)^{1/2} \overset{p}{\to} \frac{\sigma_C}{\sigma_S}.$$

We known from the previous exercise that $\widehat{\beta}_n \to_p \rho\sigma_S/\sigma_C$, so by PLIM.2 we conclude that

$$\widehat{\rho}_n \overset{p}{\to} \frac{\sigma_C}{\sigma_S}\frac{\rho\sigma_S}{\sigma_C} = \rho.$$

References

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