HOMEWORK 7
GRA6039 ECONOMETRICS WITH PROGRAMMING
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Exercise 1. (Is my coin fair?) You have a coin and you think it is a *fair* coin, that is, you think your coin is equally likely to show heads and tails. Call this your null hypothesis, the alternative being that your coin is unfair:

$$H_0\colon \text{fair coin} \quad \text{vs.} \quad H_A\colon \text{unfair coin}.$$

(a) Translate the null-hypothesis and the alternative hypothesis above into hypotheses about an unknown parameter, $p = \Pr(\text{coin showing heads})$, say.

(b) You want to test whether your coin is fair. Here is one way of going about doing that: Toss the coin 6 times and count the number of heads and tails. If you get a sequence of heads and tails, for example,

$$H\,H\,H\,T\,H\,H, \tag{1}$$

that you think is sufficiently strange for a fair coin (if my coin is fair, why just one tail?!), you might change your opinion about the coin being fair. In statistics, one says that you reject the null-hypothesis.

Suppose that H_0 is true, that is, assume that your coin is fair. What is the probability of getting the sequence in (1)? You'll find that the probability of getting (1) is small, it's a strange sequence in a way. Should you stop believing in H_0 based on this?

Look at the sequence of coin tosses

$$H\,T\,H\,T\,H\,T.$$

Still assuming H_0 is true, what is the probability of getting this sequence?

(c) Count the number of different sequences it is possible get in six tosses of a coin? *Hint*: Make a drawing.

(d) If a coin is indeed fair, a sequence like the one in (1) *is* somewhat surprising, but there are other sequences that are *just as surprising*, and still some that are *even more surprising*, given that the coin is fair. Explain why the number of sequences that are just as surprising, or even more surprising than the sequence in (1), is 14.

(e) Assuming that H_0 is true, compute the probability

$$\Pr\{\text{getting a sequence that is as surprising, or even more surprising than (1)}\}.$$

The probability you just computed is an example of a p-value. How do you feel about H_0?

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Exercise 2. Remember that the cdf of a random variable tells us all there is to know about the random variable. For example, if X has cdf F_X and Y has cdf F_Y, and it so happens that $F_X(x) = F_Y(x)$ for all x, then X and Y have the same distribution. Similarly, if X is a random variable with cdf F_X, and Y is another random variable, and g is a function such that

$$\Pr(g(Y) \leq x) = F_X(x),$$

for all x. Then the random variable $g(Y)$ has the same distribution as X. Let

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp(-y^2/2)\,\mathrm{d}y,$$

be the cdf of the standard normal distribution.

(a) Prove Lemma 1 from the lecture: If $X \sim \mathrm{N}(a, b^2)$, then

$$\frac{X-a}{b} \sim \mathrm{N}(0,1).$$

(b) If $X \sim \mathrm{N}(a, b^2)$, show that

$$\Pr(X \leq x) = \Phi\big(\frac{x-a}{b}\big).$$

(c) Let $X_1, \dots, X_n$ be i.i.d. random variables with the $\mathrm{N}(\mu, \sigma^2)$ distribution, and, as usual $\bar{X}_n = (1/n)\sum_{i=1}^{n} X_i$. Show that

$$\bar{X}_n \sim \mathrm{N}(\mu, \sigma^2/n),$$

and that

$$\bar{X}_n - \mu \sim \mathrm{N}(0, \sigma^2/n),$$

and, finally, that

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim \mathrm{N}(0,1).$$

(d) With $X_1, \dots, X_n$ as above, show that

$$\Pr(\bar{X}_n \leq x) = \Phi\big(\frac{x-\mu}{\sigma/\sqrt{n}}\big) = \Phi\big(\frac{\sqrt{n}(x-\mu)}{\sigma}\big).$$

(e) The normal distribution is symmetric around its mean: If $X \sim \mathrm{N}(\mu, \sigma^2)$, then for $z > 0$,

$$\Pr(X \leq \mu - z) = \Pr(X \geq \mu + z) = 1 - \Pr(X \leq \mu + z).$$

In particular,

$$\Pr\big(\frac{X-\mu}{\sigma} \leq -z\big) = \Phi(-z) = 1 - \Phi(z) = \Pr\big(\frac{X-\mu}{\sigma} \geq z\big).$$

Try this out in Matlab

```
z = 1.645;
normcdf(-z,0,1)
1 - normcdf(z,0,1)
norminv(normcdf(-z,0,1),0,1)
norminv(normcdf(z,0,1),0,1)
```

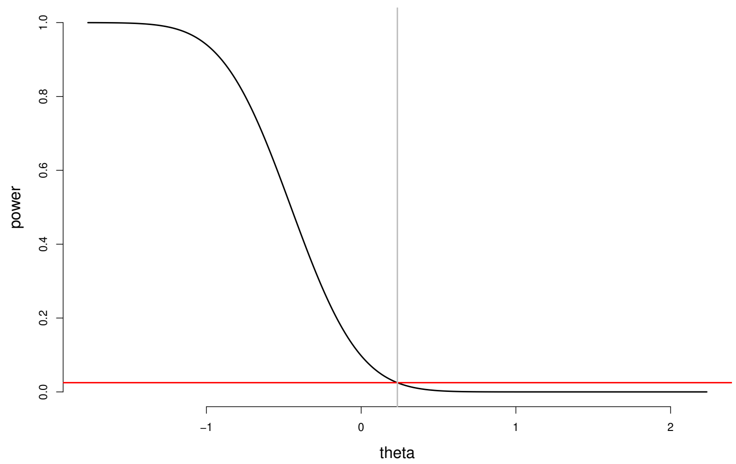

FIGURE 1. The power function in (2) for $n = 10$, $\theta_{\text{nox}} = 0.234$, and $\sigma^2 = 1.23$. The red line is the significance level 0.025. The grey line indicates θ_{nox}, to the right of this vertical line H_0 is true, to the left H_A is true.

Here `norminv(p,0,1)` is the inverse $\Phi^{-1}(p)$ of $\Phi(z)$, that is $\Phi^{-1}(\Phi(z)) = z$.

(f) Let $X_1, \ldots, X_n$ be i.i.d. $\mathrm{N}(\mu, \sigma^2)$, with $\sigma^2 = 2.34$. Find an expression for a 90% confidence interval for μ.

Exercise 3. You work for Oslo kommune and one day in the late spring your job is to check the concentration of intestinal bacteria in the water at Sørenga sjøbad. You bring $n = 10$ samples of water back to the laboratory, and measure the concentration of intestinal bacteria on a measurement device giving unbiased estimates with normally distributed errors with variance 1.23, each measurement being independent. This means that we have $X_1, \ldots, X_n$ i.i.d. random variables from a $\mathrm{N}(\theta, \sigma^2)$ distribution, with $\sigma^2 = 1.23$, and θ is the true concentration of intestinal bacteria in the water. If $\theta \geq \theta_{\text{nox}}$, with $\theta_{\text{nox}} = 0.234$, then people ought absolutely not to swim at Sørenga sjøbad. You formulate the following null- and alternative hypotheses

$$H_0\colon \theta \geq \theta_{\text{nox}}, \quad \text{vs.} \quad H_A\colon \theta < \theta_{\text{nox}}$$

(a) Explain why the hypotheses above are reasonable.

(b) It is natural to reject the null-hypothesis if $\bar{X}_n = (1/n)\sum_{i=1}^n X_i$ is much smaller than θ_{nox}. A test for the null-hypothesis is: Reject H_0 if

$$\bar{X}_n - \theta_{\text{nox}} \leq c_n,$$

for some number c_n. Explain why the c_n ensuring that the probability of rejecting H_0 when $\mathrm{E}[X_1] = \theta = \theta_{\text{nox}}$ is 0.025, is

$$c_n = \frac{\sigma\Phi^{-1}(0.025)}{\sqrt{n}} = -\frac{\sigma 1.96}{\sqrt{n}}.$$

(c) The probability of rejecting H_0 when $\mathrm{E}[X_1] = \theta$ is

$$\Pr_\theta(\text{reject } H_0) = \Pr_\theta(\bar{X}_n - \theta_{\text{nox}} \le c_n),$$

where this probability is computed for the θ given in the subscript. This means that with the c_n from (b),

$$\Pr_{\theta_{\text{nox}}}(\text{reject } H_0) = \Pr_{\theta_{\text{nox}}}(\bar{X}_n - \theta_{\text{nox}} \le c_n) = 0.025.$$

If we regard $\Pr_\theta(\text{reject } H_0)$ as a function of θ, we get what is called the power function,

$$\text{power}(\theta) = \Pr_\theta(\text{reject } H_0) = \Pr_\theta(\bar{X}_n - \theta_{\text{nox}} \le c_n). \tag{2}$$

Show that

$$\text{power}(\theta) = \Phi\big(-1.96 - \frac{\sqrt{n}(\theta - \theta_{\text{nox}})}{\sigma}\big).$$

(d) Graph the power function. Your plot should look like the plot in Figure 1.

Exercise 4. Here are $n = 10$ data points.

```
-0.1887 -0.3978  2.7470  0.4135  0.1691  1.6996  1.2608  0.1342 -0.1759 0.4977
```

You know that these are realisations of i.i.d. random variables $X_1, \ldots, X_n$ that are independent and have known variance 1. You do not know what their expectation $\mathrm{E}[X_1] = \theta$ is, and your job is to test the hypothesis that

$$H_0 \colon \theta = 0,$$

versus the alternative $H_A \colon \theta \neq 0$. Index probabilities by a subscript,

$$\Pr_\theta(X_1 \le x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\{-(y-\theta)^2/2\}\,\mathrm{d}y,$$

so that $\Pr_0(X_1 \le x) = \Phi(x)$ is the cdf of the standard normal distribution.

(a) Since $\bar{X}_n = (1/n)\sum_{i=1}^n X_i$ is an unbiased and consistent estimator of the expectation of the X_i's, it is natural to reject H_0 if

$$\bar{X}_n \le -c_n \text{ or } \bar{X}_n \ge c_n, \tag{3}$$

for some $c_n > 0$. Show that the c_n that ensures that the probability of committing a Type I error does not exceed 0.10 is

$$c_n = \frac{\Phi^{-1}(0.95)}{\sqrt{n}}.$$

where $\Phi^{-1}(p)$ is the inverse of the standard normal cdf $\Phi(z)$.

(b) The decision rule described in (3) is called a test. The power of a test is the probability that we reject H_0 for various values of the parameter we are interested in, here θ. It is the function

$$\text{power}(\theta) = \Pr_\theta(\text{reject } H_0).$$

We want power(θ) to be big when H_0 is false, and small when H_0 is true. Show that

$$\text{power}(\theta) = \Phi\{-\sqrt{n}(c_n + \theta)\} + 1 - \Phi\{\sqrt{n}(c_n - \theta)\}. \tag{4}$$

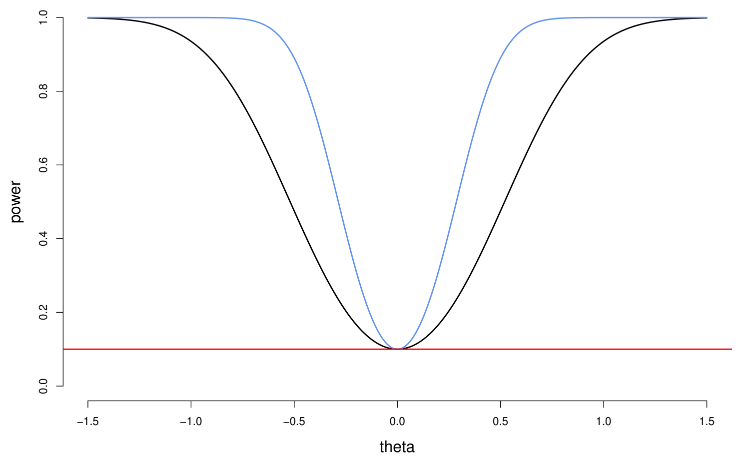

FIGURE 2. The power function in (4) for $n = 10$ and for $n = 100$. The red line is the significance level 0.10.

Hint: Use the result from Ex. 2.

(c) Reproduce the plot in Figure 2. It is a plot of the power functions when $n = 10$ and when $n = 33$.

(d) What is the probability of rejecting H_0 when $n = 10$ and the true θ equals 1/2? What is this probability when $n = 33$?

(e) Use the data given at the start of this exercise and test H_0. Conclude.

Exercise 5. (DISCRETE TIME MODEL FOR A STOCK PRICE). Suppose that S_t is the price of a stock (or some other asset), and that we observe S_t at discrete times

$$a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b.$$

With out loss of generality, we can assume that $a = 0$ and that $b = 1$. We also assume that the times between the observation times, which we denote by Δ_n, are the same for all observations times, that is,

$$\Delta_n = t_j - t_{j-1} = \frac{1}{n}, \quad \text{for } j = 1, \ldots, n,$$

so that $t_j = j/n$ for $j = 1, \ldots, n$. Let $\xi_{t_1}, \ldots, \xi_{t_n}$ be i.i.d. standard normal random variables, i.e. $\xi_i \sim \mathrm{N}(0, 1)$, and for $j = 1, \ldots, n$ define

$$Z_{t_j} = (\Delta_n)^{1/2} \sum_{i=1}^{j} \xi_{t_i}.$$

Let S_{t_j} be the price of a stock (or some asset) at time t_j. Our model for the stock price is

$$S_{t_j} = S_0 \exp(\mu t_j + \sigma Z_{t_j}), \quad \text{for } j = 1, \ldots, n, \tag{5}$$

where $\mu \in (-\infty, \infty)$ is an unknown *drift* parameter, $\sigma > 0$ is also unknown, and σ^2 is referred to as the *volatility* of the stock price, and $S_0 = S_{t_0}$ is the price of the stock at

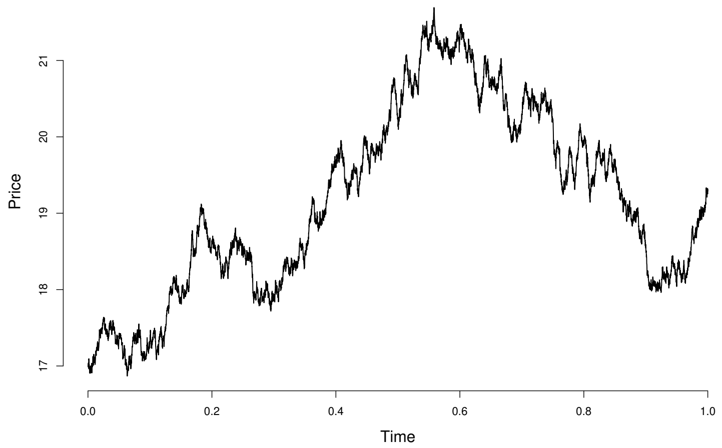

FIGURE 3. A simulated path of the price process S_{t_j} in (5) with $n = 10\,000$, $\mu = 0.123$, and $\sigma = \sqrt{0.02}$.

the start of the observation window, that we assume to be fixed and known. Define the *returns* on a stock over a one unit time interval as

$$R(t_j) = \frac{S_{t_j} - S_{t_{j-1}}}{S_{t_{j-1}}}, \quad \text{for } j = 1, \ldots, n.$$

We will also need the log-price process Y_{t_j} defined by

$$Y_{t_j} = \log(S_{t_j}), \quad \text{for } j = 0, \ldots, n.$$

(a) Set $S_0 = 17$, $\mu = 0.123$, and $\sigma = 0.02$, and simulate one path of the stock price S_{t_j} for $j = 1, \ldots, 10\,000$. The plot in Figure 3 is one example of what your plot might look like.

(b) Graph the two functions $f_1(r) = r$ and $f_2(r) = \log(1 + r)$ for $-0.99 \leq r \leq 0.99$ in one and the same plot (do this in Matlab, of course). Based on your plot, explain why

$$Y_{t_j} - Y_{t_{j-1}} \approx R(t_j), \tag{6}$$

most of the time, when t_j and t_{j-1} are not too far from each other, and the stock price is not too volatile.

(c) The approximation in (6) makes it easier to work statistically with actual stock data. Show that

$$Y_{t_j} - Y_{t_{j-1}} \sim \mathrm{N}(\mu\Delta_n, \sigma^2\Delta_n), \quad \text{for } j = 1, \ldots, n,$$

and explain why they are independent. In view of (6), this result says that the returns $R(t_1), \ldots R(t_n)$ are independent and (approximately) normally distributed. In reality they might not be, but under the model in (5), they are.

(d) A natural estimator for the drift parameter μ is

$$\widehat{\mu}_n = \frac{1}{n\Delta_n}\sum_{j=1}^{n}(Y_{t_j} - Y_{t_{j-1}}).$$

Show that $\widehat{\mu}_n$ is unbiased for μ. A problematic thing about μ is that it cannot be consistently estimated. To see that $\widehat{\mu}_n$ cannot be consistent, compute its variance.

(e) In view of (d), it is perhaps surprising that the volatility σ^2 *can* be consistently estimated. An estimator that is consistent for σ^2 is the *realised volatilty*,

$$\widehat{\sigma}_n^2 = \sum_{j=1}^{n}(Y_{t_j} - Y_{t_{j-1}})^2.$$

Write

$$\widehat{\sigma}_n^2 = \mu^2\Delta_n + 2\mu\sigma\Delta_n^{3/2}\sum_{j=1}^{n}\xi_{t_j} + \sigma^2\Delta_n\sum_{j=1}^{n}\xi_{t_j}^2,$$

and use this expression combined with Property PLIM.2 (Wooldridge (2019, p. 723) and Lemma 5.2 in the Lecture notes, in particular Remark 5.3) to show that

$$\widehat{\sigma}_n^2 \xrightarrow{p} \sigma^2, \quad \text{as } n \to \infty,$$

meaning that the realised volatility is consistent for the true volatility.

Exercise 6. (A STOCK AND AN INDEX) This exercise builds on the previous one. Suppose that for equidistant times

$$0 \leq t_0 < t_1 < \cdots < t_n = 1,$$

we observe a stock price S_{t_j} and some index C_{t_j} (the SP-500, for example). As a model for these two we take

$$S_{t_j} = S_0\exp(\mu_S t_j + \sigma_S Z_{t_j}), \quad \text{and} \quad C_{t_j} = C_0\exp(\mu_C t_j + \sigma_C W_{t_j}),$$

for $j = 1,\ldots,n$, with $\sigma_S > 0$ and $\sigma_C > 0$, where

$$Z_{t_j} = \Delta_n^{1/2}\sum_{i=1}^{j}\xi_{t_i}, \quad \text{and} \quad W_{t_j} = \Delta_n^{1/2}\sum_{i=1}^{j}\eta_i,$$

with

$$\eta_i = \rho\xi_{t_i} + (1-\rho^2)^{1/2}\epsilon_{t_i}, \quad \rho \in (-1,1), \quad \text{for } i = 1,\ldots,n,$$

and $\xi_{t_1},\ldots,\xi_{t_n},\epsilon_{t_1},\ldots,\epsilon_{t_n}$ are i.i.d. standard normal random variables. Define

$$Y_{t_j} = \log S_{t_j}, \quad \text{and} \quad X_{t_j} = \log C_{t_j}, \quad \text{for } j = 0,\ldots,n.$$

We want to say something about the relation between the stock price and the index.

(a) Show that

$$\mathrm{Cov}(\xi_{t_j},\eta_{t_j}) = \mathrm{E}\,[\xi_{t_j}\eta_{t_j}] = \rho, \quad \text{for } j = 1,\ldots,n.$$

(b) Explain why $Z_{t_j} - Z_{t_{j-1}} \sim \mathrm{N}(0,\Delta_n)$ for $j = 1,\ldots,n$, and that these are independent; and that $W_{t_j} - W_{t_{j-1}} \sim \mathrm{N}(0,\Delta_n)$ for $j = 1,\ldots,n$, and that these are independent.

(c) Show that for $j = 1, \dots, n$,

$$\mathrm{Cov}(Z_{t_j} - Z_{t_{j-1}}, W_{t_j} - W_{t_{j-1}}) = \Delta_n \rho,$$

and that

$$\mathrm{Cov}(Y_{t_j} - Y_{t_{j-1}}, X_{t_j} - X_{t_{j-1}}) = \Delta_n \sigma_S \sigma_C \rho.$$

(d) An estimator for $\sigma_S \sigma_C \rho$ is

$$\widehat{\mathrm{cov}}_n = \sum_{j=1}^{n} (X_{t_j} - X_{t_{j-1}})(Y_{t_j} - Y_{t_{j-1}}).$$

Write

$$\widehat{\mathrm{cov}}_n = \mu_S \mu_C \Delta_n + \mu_S \Delta_n^{3/2} \sum_{j=1}^{n} \eta_{t_j} + \mu_C \Delta_n^{3/2} \sum_{j=1}^{n} \xi_{t_j} + \sigma_S \sigma_C \Delta_n \sum_{j=1}^{n} \xi_{t_j} \eta_{t_j},$$

and mimic the argument from Ex. 5(e) to show that

$$\widehat{\mathrm{cov}}_n \overset{p}{\to} \sigma_S \sigma_C \rho,$$

as $n \to \infty$.

(e) Consider the function

$$g(\beta) = \sum_{j=1}^{n} \{(Y_{t_j} - Y_{t_{j-1}}) - \beta(X_{t_j} - X_{t_{j-1}})\}^2.$$

Show that the minimiser $\widehat{\beta}_n$ of this function is (the least squares estimator)

$$\widehat{\beta}_n = \frac{\sum_{j=1}^{n} (X_{t_j} - X_{t_{j-1}})(Y_{t_j} - Y_{t_{j-1}})}{\sum_{j=1}^{n} (X_{t_j} - X_{t_{j-1}})^2} = \frac{\widehat{\mathrm{cov}}_n}{\widehat{\sigma}_{C,n}^2},$$

where $\widehat{\sigma}_{C,n}^2 = \sum_{j=1}^{n} (X_{t_j} - X_{t_{j-1}})^2$.

(f) Explain why $\widehat{\beta}_n$ is consistent for $\rho \sigma_S / \sigma_C$, that is

$$\widehat{\beta}_n \overset{p}{\to} \rho \frac{\sigma_S}{\sigma_C}, \quad \text{as } n \to \infty.$$

Hint: Use the result from Ex. 5(e) as it applies to $\widehat{\sigma}_{C,n}^2$, the result from (d), and Property PLIM.2 (Lemma 5.2 in the Lecture notes).

(g) Propose an estimator, $\widehat{\rho}_n$ say, that is consistent for ρ. *Hint*: You will, I think, need both PLIM.1 and PLIM.2 to argue that your estimator is consistent.

References

Wooldridge, J. M. (2019). *Introductory Econometrics: A Modern Approach. Seventh Edition.* Cengage Learning, Boston, MA.

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