PROPOSED SOLUTIONS
HOMEWORK 6
GRA6039 ECONOMETRICS WITH PROGRAMMING
AUTUMN 2020

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Solutions to Ex. 1. The model is

$$Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i, \quad \text{for } i = 1, \ldots, n,$$

where $\varepsilon_1, \ldots, \varepsilon_n$ are independent and identically distributed (i.i.d.) random variables with expectation $\mathrm{E}[\varepsilon_1] = 0$ and variance $\mathrm{Var}(\varepsilon_1) = \sigma^2$; $x_1, \ldots, x_n$ are fixed numbers (not random variables), and we assume that they are not all equal, so that $\sum_{i=1}^n (x_i - \bar{x}_n)^2 > 0$.
(a) Use that $\mathrm{E}[a + X] = a + \mathrm{E}[X]$ when X is a rv and a is a constant,

$$\mathrm{E}[Y_i] = \mathrm{E}[\beta_0 + \beta_1 x_i + \varepsilon_i] = \beta_0 + \beta_1 x_i + \mathrm{E}[\varepsilon_i] = \beta_0 + \beta_1 x_i$$

since $\beta_0 + \beta_1 x_i$ is a constant, and $\mathrm{E}[\varepsilon_i] = 0$. Use that $\mathrm{Var}(a + X) = \mathrm{Var}(X)$ when X is a rv and a is a constant,

$$\mathrm{Var}(Y_i) = \mathrm{Var}(\beta_0 + \beta_1 x_i + \varepsilon_i) = \mathrm{Var}(\varepsilon_i) = \sigma^2.$$

(b) That $\widehat{\beta}_0$ and $\widehat{\beta}_1$ are the minimisers of $g(\beta_0, \beta_1)$ means that

$$g(\widehat{\beta}_0, \widehat{\beta}_1) \leq g(\beta_0, \beta_1) \quad \text{for all } \beta_0 \text{ and } \beta_1,$$

Then it is certainly true that $g(\widehat{\beta}_0, \widehat{\beta}_1) \leq g(1.23, 4.56)$. **(c)** Make a drawing with some data points in the plance, and think about what you find to be the 'best', or a good, line.
(d) We did this in lecture.
(e) We did this one also in lecture. It is important to remember that (See hw1, Ex. 10(a))

$$\sum_{i=1}^n (x_i - \bar{x}_n)(Y_i - \bar{Y}_n) = \sum_{i=1}^n (x_i - \bar{x}_n) Y_i.$$

(f) The expression for $\widehat{\beta}_1$ is immediate from the expression just above. For $\widehat{\beta}_0$,

$$\begin{aligned}\widehat{\beta}_0 = \bar{Y}_n - \widehat{\beta}_1 \bar{x}_n \frac{1}{n} \sum_{i=1}^n Y_i - \frac{\sum_{i=1}^n (x_i - \bar{x}_n) Y_i}{\sum_{j=1}^n (x_j - \bar{x}_n)^2} \bar{x}_n \\ = \sum_{i=1}^n \big(\frac{1}{n} - \frac{(x_i - \bar{x}_n)\bar{x}_n}{\sum_{j=1}^n (x_j - \bar{x}_n)^2}\big) Y_i.\end{aligned}$$

Date: October 6, 2020.

The reason for writing the estimators like this, is to see that they are linear in the $Y_1, \ldots, Y_n$. This fact also makes it easier to compute the expectation and the variance of the estimators. That is, we see that

$$\widehat{\beta}_1 = \sum_{i=1}^n a_i Y_i, \quad \text{and} \quad \widehat{\beta}_0 = \sum_{i=1}^n b_i Y_i, \tag{1}$$

with

$$a_i = \frac{x_i - \bar{x}_n}{\sum_{j=1}^n (x_j - \bar{x}_n)^2}, \quad \text{and} \quad b_i = \frac{1}{n} - \frac{(x_i - \bar{x}_n)\bar{x}_n}{\sum_{j=1}^n (x_j - \bar{x}_n)^2},$$

for $j = 1, \ldots, n$.

(g) We use the expression from (1). Note that $\sum_{i=1}^n a_i = 0$, and that $\sum_{i=1}^n b_i = 1$. Also,

$$\sum_{i=1}^n a_i x_i = \sum_{i=1}^n \frac{x_i - \bar{x}_n}{\sum_{j=1}^n (x_j - \bar{x}_n)^2} x_i = \frac{\sum_{i=1}^n (x_i - \bar{x}_n) x_i}{\sum_{j=1}^n (x_j - \bar{x}_n)^2} = 1,$$

because $\sum_{i=1}^n (x_i - \bar{x}_n) x_i = \sum_{i=1}^n (x_i - \bar{x}_n)^2$. Also,

$$\sum_{i=1}^n b_i x_i = \sum_{i=1}^n \Big(\frac{1}{n} - \frac{(x_i - \bar{x}_n)\bar{x}_n}{\sum_{j=1}^n (x_j - \bar{x}_n)^2}\Big) x_i = \bar{x}_n - \frac{\sum_{i=1}^n (x_i - \bar{x}_n) x_i}{\sum_{j=1}^n (x_j - \bar{x}_n)^2} \bar{x}_n = \bar{x}_n - \bar{x}_n = 0,$$

for the same reason. Then, using Prop. 2.3 in the Lecture notes,

$$\mathrm{E}\,[\widehat{\beta}_1] = \sum_{i=1}^n a_i \mathrm{E}\,[Y_i] = \sum_{i=1}^n a_i(\beta_0 + \beta_1 x_i) = \beta_1 \sum_{i=1}^n a_i x_i = \beta_1.$$

and

$$\mathrm{E}\,[\widehat{\beta}_0] = \sum_{i=1}^n b_i \mathrm{E}\,[Y_i] = \sum_{i=1}^n b_i(\beta_0 + \beta_1 x_i) = \beta_0 + \beta_1 \sum_{i=1}^n b_i x_i = \beta_0.$$

For the variance,

$$\mathrm{Var}(\widehat{\beta}_1) = \mathrm{Var}\,\Big(\sum_{i=1}^n a_i Y_i\Big) = \sum_{i=1}^n a_i^2 \mathrm{Var}(Y_i) = \sum_{i=1}^n a_i^2 \sigma^2 = \sigma^2 \sum_{i=1}^n a_i^2.$$

because the $Y_1, \ldots, Y_n$ are independent, and the a_i are constants (not random variables). See hw2, Ex. 3(d)–(f). And similarly,

$$\mathrm{Var}(\widehat{\beta}_0) = \mathrm{Var}\,\Big(\sum_{i=1}^n b_i Y_i\Big) = \sigma^2 \sum_{i=1}^n b_i^2.$$

So you only need to check that $\sum_{i=1}^n a_i^2$ and $\sum_{i=1}^n b_i^2$ are as given in the exercise.

(h) We can argue like this: We want an estimator for σ^2,

$$\mathrm{E}\,[(Y_i - \beta_0 - \beta_1 x_i)^2] = \mathrm{E}\,[\varepsilon_i^2] = \mathrm{Var}(\varepsilon_i) = \sigma^2.$$

The rv's $\varepsilon_1^2, \ldots, \varepsilon_n^2$ are i.i.d., so by the Law of large numbers their empirical mean should, for n large enough, be close to their expectation $\mathrm{E}\,[\varepsilon_1^2] = \sigma^2$. Thus,

$$\frac{1}{n} \sum_{i=1}^n \varepsilon_i^2 \approx \mathrm{E}\,[\varepsilon_i^2] = \sigma^2,$$

when n is large. We do not observe the $\varepsilon_i^2 = (Y_i - \beta_0 - \beta_1 x_i)^2$ random variables, but we hope to get close by inserting the our estimators for β_0 and β_1. This gives

$$\widehat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^n (Y_i - \widehat{\beta}_0 - \widehat{\beta}_1 x_i)^2.$$

Solutions to Ex. 2. There are many ways of doing this. Since $\widehat{\beta}_0$, β_1 solves

$$\frac{\partial}{\partial \beta_0} g(\beta_0, \beta_1) = 0, \quad \text{and} \quad \frac{\partial}{\partial \beta_1} g(\beta_0, \beta_1) = 0,$$

we know that

$$\sum_{i=1}^n (Y_i - \widehat{\beta}_0 - \widehat{\beta}_1 x_i) = 0, \quad \text{and that} \quad \sum_{i=1}^n (Y_i - \widehat{\beta}_0 - \widehat{\beta}_1 x_i) x_i = 0.$$

We use this in the fourth equality here

$$\begin{aligned}
\widehat{\sigma}_n^2 &= \frac{1}{n}\sum_{i=1}^n (Y_i - \widehat{\beta}_0 - \widehat{\beta}_1 x_i)^2 = \frac{1}{n}\sum_{i=1}^n (Y_i - \widehat{\beta}_0 - \widehat{\beta}_1 x_i)(Y_i - \widehat{\beta}_0 - \widehat{\beta}_1 x_i) \\
&= \frac{1}{n}\sum_{i=1}^n (Y_i - \widehat{\beta}_0 - \widehat{\beta}_1 x_i) Y_i - \frac{1}{n}\widehat{\beta}_0 \sum_{i=1}^n (Y_i - \widehat{\beta}_0 - \widehat{\beta}_1 x_i) - \frac{1}{n}\widehat{\beta}_1 \sum_{i=1}^n (Y_i - \widehat{\beta}_0 - \widehat{\beta}_1 x_i) x_i \\
&= \frac{1}{n}\sum_{i=1}^n (Y_i - \widehat{\beta}_0 - \widehat{\beta}_1 x_i) Y_i = \frac{1}{n}\sum_{i=1}^n Y_i^2 - \widehat{\beta}_0 \bar{Y}_n - \frac{1}{n}\widehat{\beta}_1 \sum_{i=1}^n x_i Y_i \\
&= \frac{1}{n}\sum_{i=1}^n Y_i^2 - (\bar{Y}_n - \widehat{\beta}_1 \bar{x}_n)\bar{Y}_n - \frac{1}{n}\widehat{\beta}_1 \sum_{i=1}^n (x_i - \bar{x}_n + \bar{x}_n) Y_i \\
&= \frac{1}{n}\sum_{i=1}^n Y_i^2 - \bar{Y}_n^2 + \widehat{\beta}_1 \bar{x}_n \bar{Y}_n - \frac{1}{n}\widehat{\beta}_1 \sum_{i=1}^n (x_i - \bar{x}_n) Y_i - \widehat{\beta}_1 \bar{x}_n \bar{Y}_n \\
&= \frac{1}{n}\sum_{i=1}^n Y_i^2 - \bar{Y}_n^2 - (\widehat{\beta}_1)^2 \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x}_n)^2.
\end{aligned}$$

because $\sum_{i=1}^n (x_i - \bar{x}_n) Y_i = \widehat{\beta}_1 \sum_{i=1}^n (x_i - \bar{x}_n)^2$.
(b) Using that $\mathrm{Var}(Z) = \mathrm{E}[Z^2] - (\mathrm{E}[Z])^2$ for any random variable Z, the expressions given in this exercise follow from what we found in Ex. 1.

(c) Write

$$\begin{aligned} \mathrm{E} \sum_{i=1}^n Y_i^2 = \sum_{i=1}^n \mathrm{E}\, Y_i^2 &= \sum_{i=1}^n \{\sigma^2 + (\beta_0 + \beta_1 x_i)^2\} \\ &= n\sigma^2 + \sum_{i=1}^n \{\beta_0 + \beta_1 \bar{x}_n + \beta_1(x_i - \bar{x}_n)\}^2 \\ &= n\sigma^2 + \sum_{i=1}^n \{(\beta_0 + \beta_1 \bar{x}_n)^2 + 2(\beta_0 + \beta_1\bar{x}_n)\beta_1(x_i - \bar{x}_n) + \beta_1^2(x_i - \bar{x}_n)^2\} \\ &= n\sigma^2 + n(\beta_0 + \beta_1\bar{x}_n)^2 + \beta_1^2 \sum_{i=1}^n (x_i - \bar{x}_n)^2, \end{aligned}$$

because $\sum_{i=1}^n (x_i - \bar{x}_n) = 0$.

(d) The point of finding the expression for $\widehat{\sigma}_n^2$ that we found in (a) is that it makes it easier to compute the expectation.

$$\begin{aligned} \mathrm{E}\, \widehat{\sigma}_n^2 &= \frac{1}{n}\sum_{i=1}^n \mathrm{E}\,[Y_i^2] - \mathrm{E}\,[\bar{Y}_n^2] - \frac{1}{n}\mathrm{E}\,[\widehat{\beta}_1] \sum_{i=1}^n (x_i - \bar{x}_n)^2 \\ &= \sigma^2 + (\beta_0 + \beta_1\bar{x}_n)^2 + \beta_1^2 \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x}_n)^2 \\ &\qquad - \frac{\sigma^2}{n} - (\beta_0 + \beta_1\bar{x}_n)^2 - \frac{1}{n}\Big(\frac{\sigma^2}{\sum_{j=1}^n (x_j - \bar{x}_n)2} + \beta_1^2\Big) \sum_{i=1}^n (x_i - \bar{x}_n)^2 \\ &= \sigma^2 - \frac{\sigma^2}{n} - \frac{\sigma^2}{n} = \sigma^2 \frac{n-2}{n}. \end{aligned}$$

(e) Since

$$\mathrm{E}\, \widehat{\sigma}_n^2 = \frac{n-2}{n}\sigma^2,$$

we see that an unbiased estimator of σ^2 is

$$\frac{n}{n-2}\widehat{\sigma}_n^2 = \frac{1}{n-2}\sum_{i=1}^n (Y_i - \widehat{\beta}_0 - \widehat{\beta}_1 x_i)^2.$$

Solutions to Ex. 3. The Matlab script for this exercise was written during the TA-session 30/9, and you can find the .m-files in the TA-sessions folder on Itslearning. For exercise (d) and (e), use techniques similar to those used in Ex. 1.

Solutions to Ex. 4. The Matlab script for this exercise is also in the the TA-sessions folder on Itslearning, and is called `TA_session6_Ex4.m`.

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