PROPOSED SOLUTIONS
HOMEWORK 5
GRA6039 ECONOMETRICS WITH PROGRAMMING
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EMIL A. STOLTENBERG

Solutions to Ex. 1. Assume that $X_n \to_p a$ and $Y_n \to_p b$ as $n \to \infty$.
(a) Let c be a constant not equal to zero (or else it is trivial). For any $\varepsilon > 0$,

$$\Pr(|cX_n - ca| \geq \varepsilon) = \Pr(|c||X_n - a| \geq \varepsilon) = \Pr(|X_n - a| \geq \varepsilon/|c|) \to 0,$$

because $\Pr(|X_n - a| \geq \varepsilon) \to 0$ for any $\varepsilon > 0$. Thus $cX_n \xrightarrow{p} ca$.
(b) That

$$X_n + Y_n \xrightarrow{p} a + b,$$

is shown in the Lecture notes, under Lecture 5.
(c) To show that $X_nY_n \to_p ab$, write

$$X_nY_n - ab = (X_n - a)(Y_n - b) + b(X_n - a) + a(Y_n - b).$$

From (a) we know that both $b(X_n - a) \to_p 0$ and that $a(Y_n - b) \to_p 0$, so because of the result in (b) it suffices to show that $(X_n - a)(Y_n - b) \to_p 0$. Since both $X_n - a \to_p 0$ and $Y_n - b \to_p 0$ we can assume that $a = b = 0$. You then 'only' need to show that for any $\varepsilon > 0$, $\Pr(|X_nY_n| \geq 0) \to 0$. To show this, mimic the proof of (b).
(d) To show that $X_n/Y_n \to_p a/b$ provided $b \neq 0$, we can argue as follows. Set $Z_n = 1/Y_n$. Then $Z_n \to_p 1/b$ by Property PLIM.1. But then $X_n/Y_n = X_nZ_n \to_p a/b$ by (c).

Solutions to Ex. 2. A random variable Y has the Poisson distribution with parameter θ if its pmf is

$$f_\theta(y) = \frac{1}{y!}\theta^y \exp(-\theta), \quad \text{for } y = 0, 1, 2, \ldots,$$

and $f_\theta(y) = 0$ otherwise. Let $Y_1, \ldots, Y_n$ be i.i.d. Poisson with parameter θ. You might need the following fact about maximum likelihood estimators: If $\widehat{\theta}_n$ is the MLE for θ, and $h(x)$ is some real valued function, then $h(\widehat{\theta}_n)$ is the MLE for $h(\theta)$.
(a) The MLE for θ is $\widehat{\theta}_n = \bar{Y}_n = (1/n)\sum_{i=1}^n Y_i$ (see hw3 Ex. 1). Since the $Y_1, \ldots, Y_n$ are i.i.d., and $\mathrm{E}\,[Y_1] = \theta$ and $\mathrm{Var}(Y_1) = \theta$, it follows from the Law of large numbers (Theorem 4.3 in the Lecture notes) that $\bar{Y}_n \to_p \mathrm{E}\,[Y_1] = \theta$. So $\widehat{\theta}_n = \bar{Y}_n$ is consistent for θ.
(b) $\Pr_\theta(Y = 0) = f_\theta(0) = \exp(-\theta)$. Using the fact about MLE's given at the start of exercise, the MLE for $\alpha = \Pr_\theta(Y = 0) = \exp(-\theta)$ is $\widehat{\alpha}_n = \exp(-\widehat{\theta}_n)$.

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(c) The function $g(x) = \exp(-x)$ is continuous, so $\widehat{\alpha}_n = \exp(-\widehat{\theta}_n) \to_p \exp(-\theta) = \alpha$, so $\widehat{\alpha}_n$ is consistent for α.

Solutions to Ex. 3. Two random variables with the same cdf have the same distribution. In a previous homework we met the Pareto distribution. Recall that the pdf of the Pareto distribution is

$$f_\alpha(x) = \alpha \frac{x_{\min}^\alpha}{x^{\alpha+1}}, \quad \text{for } x \geq x_{\min},$$

and $f_\alpha(x) = 0$ when $x < x_{\min}$, where $x_{\min} > 0$ and $\alpha > 0$. In this exercise, we assume that $x_{\min}$ is known, while α is a parameter we want to estimate. Suppose that $X_1, \dots, X_n$ are i.i.d. from $f_\alpha(x)$. If you have not already done so, please show that the maximum likelihood estimator for α is

$$\widehat{\alpha}_n = \frac{n}{\sum_{i=1}^n \log(X_i/x_{\min})}.$$

We will now show that $\widehat{\alpha}_n$ is consistent for α, i.e. that $\widehat{\alpha}_n \to_p \alpha$.
(a) Let $x \geq x_{\min}$ then,

$$F_\alpha(x) = \int_{x_{\min}}^x \frac{x_{\min}^\alpha}{y^{\alpha+1}} \,\mathrm{d}y = -\Big|_{x_{\min}}^x \frac{x_{\min}^\alpha}{y^\alpha} = 1 - \big(\frac{x_{\min}}{x}\big)^\alpha,$$

and $F_\alpha(x) = 0$ for $x < x_{\min}$.
(b) Suppose $X \sim F_\alpha$, and set

$$Y = \log(X/x_{\min}).$$

To find the distribution of Y we find the cdf of Y. First note that Y takes its values in $[0, \infty)$. For $y \geq 0$

$$\begin{aligned} F_Y(y) &= \Pr(Y \leq y) = \Pr(\log(X/x_{\min}) \leq y) = \Pr\{(X/x_{\min}) \leq \exp(y)\} \\ &= \Pr\{X \leq x_{\min}\exp(y)\} = F_\alpha(x_{\min}\exp(y)) = 1 - \exp(-\alpha y), \end{aligned}$$

and $F_Y(y) = 0$ for $y < 0$.
(c) We recognise $F_Y(y)$ as the cdf of the exponential distribution with parameter α. So from hw2 Ex. 5, we have that

$$\mathrm{E}[Y] = \frac{1}{\alpha}, \quad \text{and} \quad \mathrm{Var}(Y) = \frac{1}{\alpha^2}.$$

(d) Let $Y_i = \log(X_i/x_{\min})$ for $i = 1, \dots, n$. Since the $X_1, \dots, X_n$ are i.i.d., the $Y_1, \dots, Y_n$ are i.i.d. exponentials with expectation $1/\alpha$ and variance $1/\alpha^2 < \infty$. It then follows from the Law of large numbers that $\bar{Y}_n = (1/n)\sum_{i=1}^n Y_i \to_p 1/\alpha$.
(e) To see that $\widehat{\alpha}_n$ is consistent for α, we write

$$\widehat{\alpha}_n = \frac{n}{\sum_{i=1}^n \log(X_i/x_{\min})} = \frac{n}{\sum_{i=1}^n Y_i} = \frac{1}{\bar{Y}_n}.$$

In (c) we saw that $\bar{Y}_n \to_p 1/\alpha$, but $g(x) = 1/x$ is a continuous function, so from Property PLIM.2, we have that $\widehat{\alpha}_n = 1/\bar{Y}_n \to 1/(1/\alpha) = \alpha$.

Solutions to Ex. 4. Suppose that $X_1, \dots, X_n$ are i.i.d. random variables with the uniform distribution on $[0, \theta]$, i.e. its pdf is $f(x) = 1/\theta$ for $0 \le x \le \theta$ and zero otherwise. Let $M_n = \max_{i \le n} X_i = \max\{X_1, \dots, X_n\}$ be the largest of the observations. In this exercise you need to know that

$$\lim_{n\to\infty} (1 + x/n)^n = \exp(x),$$

for any real number x.

(a) The cdf of one uniform X on $[0, \theta]$ is

$$F(x) = \begin{cases} 0, & \text{for } x < 0, \\ x/\theta, & \text{for } 0 \le x \le \theta, \\ 1 & \text{for } \theta \le x. \end{cases} \tag{1}$$

See the solution to Ex. 7 in hw2.

(b) That

$$\Pr(M_n \le x) = F(x)^n.$$

is also in the solution to Ex. 7 in hw2.

(c) Consider the sequence $(Z_n)_{n\ge1}$ of random variables defined by $Z_n = n(\theta - M_n)$ for $n = 1, 2, \dots$. Its cdf $F_n(z)$ is

$$\begin{aligned} F_n(z) &= \Pr(Z_n \le z) = \Pr\{n(\theta - M_n) \le z\} = \Pr\{\theta - M_n \le z/n\} \\ &= \Pr\{-M_n \le z/n - \theta\} = \Pr\{M_n \ge \theta - z/n\} = 1 - \Pr\{M_n \le \theta - z/n\} \\ &= 1 - F(\theta - z/n)^n = 1 - \big(1 - \frac{z}{\theta n}\big)^n. \end{aligned}$$

for $z \ge 0$ and $F_n(z) = 0$ for $z < 0$.

(d) To show that Z_n converges in distribution we must show that $F_n(z)$ converges to some cdf. For $z > 0$

$$F_n(z) = 1 - \big(1 - \frac{z}{\theta n}\big)^n \to 1 - \exp(-z/\theta).$$

as $n \to \infty$. But the expression on the right is the cdf of an exponential distribution with expectation θ and variance θ^2 (see hw2 Ex. 5). Thus, $Z_n \to_d Z$, where Z is a random variable with pdf $f_{1/\theta}(z) = (1/\theta)\exp(-z/\theta)$ for $z > 0$, and $\mathrm{E}[Z] = \theta$, and $\mathrm{Var}(Z) = \theta^2$.

Solutions to Ex. 5. Let $Y_1, \dots, Y_n$ be i.i.d. random variables with $\mathrm{E}[Y_1] = \theta$ and $\mathrm{Var}(Y_1) = \tau^2 = 2.34$, which means that the expectation is unknown while the variance is known. A natural estimator for θ is $\widehat{\theta}_n = \bar{Y}_n = n^{-1}\sum_{i=1}^n Y_i$. Two reasons for this being a natural estimator are that it is unbiased for θ, and that it is consistent for θ.

(a) Since the $Y_1, \dots, Y_n$ are i.i.d., $\mathrm{E}[Y_1] = \theta$ and $\mathrm{Var}(Y_1) = \tau^2$, it follows from the Central limit theorem (CLT) (Theorem 5.4 in the Lecture notes) that

$$\sqrt{n}(\widehat{\theta}_n - \theta)/\tau \xrightarrow{d} Z,$$

where $Z \sim \mathrm{N}(0, 1)$, i.e. is a standard normal random variable.

(b) First, run

```
norminv(0.95) % = 1.6449
norminv(0.05) % = -1.6449
```

which tells us that

$$\Pr(-1.6449 \leq Z \leq 1.6449) = \Phi(1.6449) - \Phi(-1.6449) = 0.90.$$

See hw1 Ex. 11 for the first equality. From the CLT we know that

$$\Pr(-1.6449 \leq \sqrt{n}(\widehat{\theta}_n - \theta)/\tau \leq 1.6449) \approx \Pr(-1.6449 \leq Z \leq 1.6449) = 0.90, \quad (2)$$

so after rearranging,

$$\Pr(\widehat{\theta}_n - 1.6449\frac{\tau}{\sqrt{n}} \leq \theta\tau \leq \widehat{\theta}_n + 1.6449\frac{\tau}{\sqrt{n}}) \approx= 0.90.$$

The lower bound is therefore $\widehat{\theta}_n - 1.6449\tau/\sqrt{n}$ and the upper bound is $\widehat{\theta}_n + 1.6449\tau/\sqrt{n}$. A realisation of the interval

$$[\widehat{\theta}_n - 1.6449\tau/\sqrt{n}, \widehat{\theta}_n + 1.6449\tau/\sqrt{n}],$$

is a 90% confidence interval.
(c) The data in `hw5_data.txt` gives the interval $[3.1723, 3.4239]$. Here is the Matlab code

```
cd("your path");
data = readmatrix("hw5_data.txt");
y = data(:,1);
n = length(y);
thetahat = mean(y);
LL = thetahat + sqrt(2.34/n)*norminv(0.05)
UU = thetahat + sqrt(2.34/n)*norminv(0.95)
```

(d) Here is the Matlab code. Run it a few times and see how `mean(contains)` is always close to 0.90. Try it out for smaller and larger sample sizes n, i.e. change `n = 50`. The approximation in (2) becomes better with larger n, try to see how `mean(contains)` might jump more from simulation to simulation when n is small, while it is very close to 0.90 in each simulation when n is big.

```
a = 2*2.34;
b = 1/sqrt(2);
n = 50;
sims = 1000
contains = zeros(1,sims);
for i = 1:sims
    y = gamrnd(a,b,1,n);
    Ln = mean(y) + sqrt(a*b^2/n)*norminv(0.05)
    Un = mean(y) + sqrt(a*b^2/n)*norminv(0.95)

    contains(i) = 1*((Ln <= a*b)&(a*b <= Un));
end

mean(contains)
```

Solutions to Ex. 5. Simulate a conditional probability. Consider a fair die: Its possible outcomes are $\{1, 2, 3, 4, 5, 6\}$, and it being fair means that $\Pr(X = j) = 1/6$ for $j = 1, 2, 3, 4, 5, 6$, where X is the random variable we associate with a roll of a die. Let $A = \{1, 2, 3\}$, and $B = \{1, 3, 5\}$ be two events.
(a) $A \cap B = \{1, 3\}$.

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} \frac{\Pr(\{1, 3\})}{\Pr(\{1, 3, 5\})} = \frac{2/6}{3/6} = \frac{2}{3},$$

where we use the definition of conditional probability, and that the die is fair. **(b)** In the script below we use simulations to check the answer from (a). In other words, we *estimate* $\Pr(A \mid B)$ on simulated data. Understand the code, run it a few times, and check you answer from (a).

```
x = datasample(1:6,100,"Replace",true);
inA = zeros(1,length(x)); inB = zeros(1,length(x));
for i = 1:length(x)
    if x(i) <= 3
        inA(i) = 1;
    end
    if (x(i) == 1)|(x(i) == 3)|(x(i) == 5)
        inB(i) = 1;
    end
end

mean(inA.*inB)/mean(inB)
```

The function `datasample` samples from the numbers $\{1, 2, 3, 4, 5, 6\}$ with uniform probability. The symbol | reads ‘or’, as in the union of two sets.

Department of Economics, BI Norwegian Business School
Email address: emil.a.stoltenberg@bi.no