PROPOSED SOLUTIONS
HOMEWORK 1
GRA6039 ECONOMETRICS WITH PROGRAMMING
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The definitions, propositions, etc. that I refer to are those in the Lecture notes. If you spot mistakes in the proposed solutions, please send me an email.

Solution to Ex. 3. **(d)** Recall that $B \setminus A = B \cap A^c$. Write $B = (B \cap A) \cup (B \setminus A) =$ (make a Venn diagram!), and notice that $B \cap A$ and $B \cap A^c$ are disjoint. Use Def. 1.2(iii) in the Lecture notes,

$$\Pr(B) = \Pr((B \cap A) \cup (B \setminus A)) = \Pr(B \cap A) + \Pr(A \setminus B).$$

Subtract $\Pr(B \cap A)$ on both sides. **(e)** Write

$$A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B),$$

and note that all three events on the right are disjoint. Use Def. 1.2(iii) and (d),

$$\begin{aligned}\Pr(A \cup B) &= \Pr(A \setminus B) + \Pr(B \setminus A) + \Pr(A \cap B)\\ &= \{\Pr(A) - \Pr(A \cap B)\} + \{\Pr(B) - \Pr(A \cap B)\} + \Pr(A \cap B)\\ &= \Pr(A) + \Pr(B) - \Pr(A \cap B).\end{aligned}$$

(f) Assume that $A \subset B$. Then $A \cap B = A$, so

$$\Pr(B \setminus A) = \Pr(B) - \Pr(A \cap B) = \Pr(B) - \Pr(A),$$

but $\Pr(B \setminus A) \geq 0$, so $\Pr(A) \leq \Pr(B)$.

Solutions to Ex. 4. **(a)** From the definition of conditional probability, multiplying both sides by $\Pr(B)$ we get $\Pr(A \cap B) = \Pr(A \mid B)\Pr(B)$. Here we can think of zero divided by zero as zero, so that if $\Pr(B) = 0$ this relation is still valid. To see that this is ok, note that since $A \cap B \subset B$, so that $\Pr(B) = 0$ entails that $\Pr(A \cap B) = 0$ (see Ex. 3(f)). By the symmetry $\Pr(A \cap B) = \Pr(B \cap A)$, so $\Pr(A \cap B) = \Pr(B \mid A)\Pr(A)$.

(b) The events $A_1, \ldots, A_k$ are pairwise disjoint and their union equals the sample space Ω. Therefore (make a drawing!),

$$B = (B \cap A_1) \cup (B \cap A_2) \cup \cdots \cup (B \cap A_{k-1}) \cup (B \cap A_k) = \bigcup_{i=1}^{k} (B \cap A_i)$$

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Since the A_is are pairwise disjoint, so are the $(B \cap A_i)$s, that is $(B \cap A_i) \cap (B \cap A_j) = \emptyset$, whenever $i \neq j$. Using Def. 1.2(iii) and (a) we then get

$$\Pr(B) = \Pr(\bigcup_{i=1}^{k}(B \cap A_i)) = \sum_{i=1}^{k} \Pr(B \cap A_i) = \sum_{i=1}^{k} \Pr(B \mid A_i)\Pr(A_i).$$

(c) Combining the definition of conditional probability with (a), we get Bayes theorem.

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\Pr(B \mid A)\Pr(A)}{\Pr(B)}.$$

(d) Combine (b) and (c).

(e) Define $R = \{\text{car is red}\}$ and $W = \{\text{witness says car was red}\}$. We are being told that $\Pr(R) = 1/10^3$. The witness can say the car is red when it is (the event R), and when it is not (the event R^c), therefore,

$$\begin{aligned}\Pr(W) &= \Pr(W \mid R)\Pr(R) + \Pr(W \mid R^c)\Pr(R^c)\\ &= \Pr(W \mid R)\Pr(R) + \Pr(W \mid R^c)\{1 - \Pr(R^c)\}.\end{aligned}$$

We also know from the psychologist that $\Pr(W \mid R) = 1$, and that $\Pr(W \mid R^c) = 5/100$. Use Bayes theorem,

$$\begin{aligned}\Pr(R \mid W) &= \frac{\Pr(W \mid R)\Pr(R)}{\Pr(W \mid R)\Pr(R) + \Pr(W \mid R^c)\Pr(R^c)}\\ &= \frac{1/10^3}{1/10^3 + 5/100 \times 999/1000} = \frac{100}{5095} \approx 0.02.\end{aligned}$$

Solution to Ex. 5. (a)

$$\sum_{n=1}^{6} 2^n = 2 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 = 2 + 4 + 8 + 16 + 32 + 64 = 126.$$

(b)

$$\begin{aligned}\sum_{k=1}^{5}(3k - 2) &= (3 \times 1 - 2) + (3 \times 2 - 2) + (3 \times 3 - 2) + (3 \times 4 - 2) + (3 \times 5 - 2)\\ &= 1 + 4 + 7 + 10 + 13 = 35.\end{aligned}$$

(c)

$$\begin{aligned}\sum_{n=0}^{5} 2/(n+1) &= 2/(0+1) + 2/(1+1) + 2/(2+1) + 2/(3+1) + 2/(4+1) + 2/(5+1)\\ &= 2 + 1 + 2/3 + 1/2 + 2/5 + 1/3 = 4 + 9/10 = 4.9.\end{aligned}$$

(d) For the sum $\sum_{i=1}^{3}(\sum_{j=1}^{3} 2^{i+j})$ notice that for $i = 1, 2, 3$, the inner sum in

$$\sum_{j=1}^{3} 2^{i+j} = 2^{i+1} + 2^{i+2} + 2^{i+3} = 2^i(2 + 2^2 + 2^3) = 2^i(2 + 4 + 8) = 2^i \times 14.$$

Therefore,

$$\sum_{i=1}^{3}(\sum_{j=1}^{3} 2^{i+j}) = \sum_{i=1}^{3} 2^i 14 = 2^1 + 2^2 14 + 2^3 14 =$$
$$(2 + 2^2 + 2^3)14 = (14)14 = 196.$$

Solution to Ex. 6. To show that the equality is not true, it suffices to find one counterexample. Take $n = 2$, and suppose that $X_1 = 1$ and $X_2 = -1$. Then

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 = \frac{1}{2}(1+1) = 1 \neq \big(\frac{1}{n}\sum_{i=1}^{n} X_i\big)^2 = \big(\frac{1}{2}(1-1)\big)^2 = 0.$$

Solution to Ex. 7. (a)

$$\sum_{i=1}^{n}(X_i + Y_i) = (X_1 + Y_1) + \cdots + (X_n + Y_n)$$
$$= X_1 + \cdots + X_n + Y_1 + \cdots + Y_n = \sum_{i=1}^{n} X_i + \sum_{i=1}^{n} Y_i.$$

(b) When a is a constant,

$$\sum_{i=1}^{n} aX_i = aX_1 + \cdots + aX_n = a(X_1 + \cdots + X_n) = a\sum_{i=1}^{n} X_i.$$

(c) In (a), set $Y_i = a$ for $i = 1, \ldots, n$, then that exercise tells us that

$$\sum_{i=1}^{n}(X_i + a) = \sum_{i=1}^{n} X_i + \sum_{i=1}^{n} a = \sum_{i=1}^{n} X_i + na.$$

(d) In (b) set $a = 1/b$ for $b \neq 0$, then $\sum_{i=1}^{n} X_i/b = (1/b)\sum_{i=1}^{n} X_i$.

Solution to Ex. 8. (a) Combining Ex. 7(a) and (b), we get

$$\sum_{i=1}^{n}(aX_i + bY_i) \overset{\text{Ex. 7(a)}}{=} \sum_{i=1}^{n} aX_i + \sum_{i=1}^{n} bY_i \overset{\text{Ex. 7(b)}}{=} a\sum_{i=1}^{n} X_i + b\sum_{i=1}^{n} Y_i.$$

(b) $A_i = aX_i$ for $i = 1, \ldots, n$, and $\bar{A}_n = (1/n)\sum_{i=1}^{n} A_i$. Then

$$\bar{A}_n = \frac{1}{n}\sum_{i=1}^{n} A_i = \frac{1}{n}\sum_{i=1}^{n} aX_i \overset{\text{Ex. 7(b)}}{=} \frac{a}{n}\sum_{i=1}^{n} X_i = a\bar{X}_n,$$

where $\bar{X}_n = (1/n)\sum_{i=1}^{n} X_i$.

(c) $B_i = X_i + Y_i$ for $i = 1, \ldots, n$. Then

$$\bar{B}_n = \frac{1}{n}\sum_{i=1}^{n} B_i = \frac{1}{n}\sum_{i=1}^{n}(X_i + Y_i) \overset{\text{Ex. 7(a)}}{=} \frac{1}{n}\sum_{i=1}^{n} X_i + \frac{1}{n}\sum_{i=1}^{n} Y_i = \bar{X}_n + \bar{Y}_n,$$

where $\bar{X}_n = (1/n)\sum_{i=1}^{n} X_i$ and $\bar{Y}_n = (1/n)\sum_{i=1}^{n} Y_i$.

(d) $C_i = aX_i + bY_i$ for $i = 1, \dots, n$. Then

$$\bar{C}_n = \frac{1}{n}\sum_{i=1}^n C_i = \frac{1}{n}\sum_{i=1}^n (aX_i + bY_i) \overset{\text{Ex. 8(a)}}{=} \frac{a}{n}\sum_{i=1}^n X_i + \frac{b}{n}\sum_{i=1}^n Y_i = a\bar{X}_n + b\bar{Y}_n.$$

Solution to Ex. 9. **(a)** Let $X_1, \dots, X_n$ be numbers, obs., rv's.

(a) We define $\widetilde{X}_i = X_i - \bar{X}_n$ for $i = 1, \dots, n$, where $\bar{X}_n = (1/n)\sum_{i=1}^n X_i$ as always. The average of $\widetilde{X}_1, \dots, \widetilde{X}_n$ is

$$\frac{1}{n}\sum_{i=1}^n \widetilde{X}_i = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X}_n) \overset{\text{Ex. 7(c)}}{=} \frac{1}{n}\sum_{i=1}^n X_i - \frac{n}{n}\bar{X}_n = \bar{X}_n - \bar{X}_n = 0.$$

(b) Now set

$$\widetilde{X}_i = \frac{X_i - \bar{X}_n}{s_X}, \quad \text{for } i = 1, \dots, n,$$

where $s_X = \sqrt{s_X^2}$, and

$$s_X^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2,$$

is the empirical variance. The empirical average of $\widetilde{X}_1, \dots, \widetilde{X}_n$ is zero:

$$\frac{1}{n}\sum_{i=1}^n \widetilde{X}_i = \frac{1}{n}\sum_{i=1}^n \frac{X_i - \bar{X}_n}{s_X} \overset{\text{Ex. 7(d)}}{=} \frac{1}{s_X}\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X}_n) \overset{\text{Ex. 9(a)}}{=} 0,$$

because $1/s_X$ is a constant.

The empirical variance $s_{\widetilde{X}}^2$ of $\widetilde{X}_1, \dots, \widetilde{X}_n$ is one:

$$\begin{aligned} s_{\widetilde{X}}^2 &= \frac{1}{n-1}\sum_{i=1}^n \big(\widetilde{X}_i - \frac{1}{n}\sum_{i=1}^n \widetilde{X}_i\big)^2 = \frac{1}{n-1}\sum_{i=1}^n \widetilde{X}_i^2 \\ &= \frac{1}{n-1}\sum_{i=1}^n \big(\frac{X_i - \bar{X}_n}{s_X}\big)^2 = \frac{1}{s_X^2}\frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2 = \frac{s_X^2}{s_X^2} = 1, \end{aligned}$$

so the empirical standard deviation $s_{\widetilde{X}} = \sqrt{s_{\widetilde{X}}^2}$ must also be one.

(c) Now set $\widetilde{X}_i = aX_i + b$ for $i = 1, \dots, n$. Then the empirical average of $\widetilde{X}_1, \dots, \widetilde{X}_n$ is

$$\frac{1}{n}\sum_{i=1}^n \widetilde{X}_i = \frac{1}{n}\sum_{i=1}^n (aX_i + b) \overset{\text{Ex. 7(c)}}{=} \frac{a}{n}\sum_{i=1}^n X_i + b = a\bar{X}_n + b.$$

And

$$\begin{aligned} s_{\widetilde{X}}^2 &= \frac{1}{n-1}\sum_{i=1}^n \{aX_i + b - (a\bar{X}_n + b)\}^2 = \frac{1}{n-1}\sum_{i=1}^n (aX_i - a\bar{X}_n)^2 \\ &\overset{\text{Ex. 7(b)}}{=} \frac{a^2}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2 = a^2 s_X^2. \end{aligned}$$

The result in (a) is the special case where $a = 1$ and $b = -\bar{X}_n$. The result in (b) is the special case where $a = 1/s_X$ and $b = -\bar{X}_n/s_X$.

Solution to Ex. 10. We have numbers, observations, rv's $X_1,\dots,X_n$ and $Y_1,\dots,Y_n$. The empirical covariance is defined as

$$s_{X,Y} = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n)(Y_i - \bar{Y}_n).$$

(a)

$$\begin{aligned} s_{X,Y} &= \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n)(Y_i - \bar{Y}_n) = \frac{1}{n-1}\sum_{i=1}^n \{(X_i - \bar{X}_n)Y_i - (X_i - \bar{X}_n)\bar{Y}_n\} \\ &\overset{\text{Ex. 7(a)}}{=} \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n)Y_i - \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n)\bar{Y}_n \\ &\overset{\text{Ex. 7(b)}}{=} \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n)Y_i - \frac{\bar{Y}_n}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n) = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n)Y_i. \end{aligned}$$

because $\sum_{i=1}^n (X_i - \bar{X}_n) = 0$ as shown in Ex. 9(a). By the same argument we see that $s_{X,Y} = (1/(n-1))\sum_{i=1}^n (Y_i - \bar{Y}_n)X_i$.

(b) From (a) and using that $s_{X,X} = s_X^2$, we get

$$\begin{aligned} s_X^2 &= \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n)X_i = \frac{1}{n-1}\sum_{i=1}^n (X_i^2 - \bar{X}_n X_i) \\ &\overset{\text{Ex. 7(a)}}{=} \frac{1}{n-1}\sum_{i=1}^n X_i^2 - \frac{1}{n-1}\sum_{i=1}^n \bar{X}_n X_i \overset{\text{Ex. 7(b)}}{=} \frac{1}{n-1}\sum_{i=1}^n X_i^2 - \bar{X}_n \frac{1}{n-1}\sum_{i=1}^n X_i \\ &= \frac{1}{n-1}\sum_{i=1}^n X_i^2 - \frac{n}{n-1}\bar{X}_n^2, \end{aligned}$$

for the last equality using that since

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i, \quad \text{we have} \quad n\bar{X}_n = \sum_{i=1}^n X_i.$$

Solution to Ex. 11. For $a < b$, define $A = \{X \le a\}$ and $B = \{a < X \le b\}$. **(a)** Then $A \cap B = \emptyset$, so they are disjoint, and **(b)** $A \cup B = \{X \le b\}$. **(c)**. Use Def. 1.2(iii)

$$\Pr(X \le b) = \Pr(A \cup B) = \Pr(A) + \Pr(B) = \Pr(X \le a) + \Pr(a < X \le b),$$

and subtract $\Pr(X \le a)$ on both sides to get $\Pr(a < X \le b) = \Pr(X \le b) - \Pr(X \le a)$.

You could also argue like this: When $a < b$,

$$\{a < X \le b\} = \{X \le b\} \setminus \{X \le a\} = \{X \le b\} \cap \{X > a\}^c,$$

and $\{X \le b\} \cap \{X \le a\} = \{X \le a\}$, so by Ex. 3(d),

$$\Pr(a < X \le b) = \Pr(\{X \le b\} \setminus \{X \le a\}) = \Pr(X \le b) - \Pr(X \le a).$$

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