# HOMEWORK 1
## GRA6039 ECONOMETRICS WITH PROGRAMMING
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**Exercise 1.** (If you are a native English speaker, you might skip this assignment.) Please start reading an English book. You should make time for reading English books, even if your schedule appears full.

*Pragmatic motivation for the short-run:* All instruction in your degree will from now on be in English, and you will write your home-exam and, eventually, your master's thesis in English.

*Pragmatic motivation for the long-run:* In your future job, you will most likely use English daily, perhaps even as the official language of your company. English will therefore be important also for progressing in your career. And while "basic English" suffice for basic needs, you will in the long run benefit tremendously from knowing much better English than that.

**Exercise 2.** Install Matlab on your own computer. Follow the instructions on `https://web.bi.no/info/swdl.nsf/ProfileListing.xsp`.

Go to `https://matlabacademy.mathworks.com`, and work through Section 1 & 2 in the course MATLAB for financial applications. Follow the instructions already given on Itslearning for the registration process.

**Exercise 3.** Let $A$ and $B$ be events in $\Omega$, and Pr a probability function. When all we know about Pr is that it obeys Kolmogorov's axioms (see Def. 1.2 in the Lecture notes), then the properties in Proposition 1.3 in the Lecture notes must be proven.

Show that

(d) $\Pr(B \setminus A) = \Pr(B) - \Pr(A \cap B)$. *Hint*: Write $B = (B \cap A) \cup (B \cap A^c)$.
(e) $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$.
(f) If $A \subset B$, then $\Pr(A) \leq \Pr(B)$.

**Exercise 4.** Recall that if $A$ and $B$ are events, and $\Pr(B) > 0$, then the conditional probability of $A$ given $B$ is defined as

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}.$$

(a) Show that $\Pr(A \cap B) = \Pr(A \mid B)\Pr(B) = \Pr(B \mid A)\Pr(A)$.

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(b) Suppose that $A_1, \dots, A_k$ are set such that $A_i \cap A_j = \emptyset$ for all $j \neq i$, and that $A_1 \cup A_2 \cup \cdots \cup A_{k-1} \cup A_k = \Omega$. Show that

$$\Pr(B) = \Pr(B \mid A_1)\Pr(A_1) + \cdots + \Pr(B \mid A_k)\Pr(A_k).$$

This is known as the Law of total probability.

(c) Let $A$ and $B$ be events with $\Pr(B) > 0$, show that

$$\Pr(A \mid B) = \frac{\Pr(B \mid A)\Pr(A)}{\Pr(B)}.$$

This is Bayes theorem.

(d) Suppose that $A_1, \dots, A_k$ are set such that $A_i \cap A_j = \emptyset$ for all $j \neq i$, and that $A_1 \cup A_2 \cup \cdots \cup A_{k-1} \cup A_k = \Omega$. Show that

$$\Pr(A_i \mid B) = \frac{\Pr(B \mid A_i)\Pr(A_i)}{\sum_{j=1}^{k} \Pr(B \mid A_i)\Pr(A_i)}.$$

(e) (Eide, 2020). Assume that a traffic accident has taken place on an island without access to the main land (cars can't get off the island). The car involved in the accident sped off and disappeared. To assess the colour of the car, the judge has the following information. (1) A witness says the car involved in the accident was red. (2) A psychologist says that witnesses are wrong in 5% of such situations in the sense that they say the car was red when in fact the car is not red. Moreover, if the car is indeed red, witnesses always say the car was red. (3) Of all the cars on the island, 0.1% are red.

What is the probability that the car involved in the accident was red?

**Exercise 5.** Calculate the following sums by hand.

(a) $\sum_{n=1}^{6} 2^n$.
(b) $\sum_{k=1}^{5} (3k-2)$.
(c) $\sum_{n=0}^{5} 2/(n+1)$.
(d) $\sum_{i=1}^{3} \big(\sum_{j=1}^{3} 2^{i+j}\big)$.

**Exercise 6.** Is it true that

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 = \big(\frac{1}{n}\sum_{i=1}^{n} X_i\big)^2, \tag{1}$$

for any numbers $X_1, X_2, \dots, X_n$? Find a simple counter-example which shows that this is not true. Suppose $X_1, X_2, \dots, X_n$ are heights of $n$ men. Explain what the left hand side of eq. (1) means, what the right hand side of eq. (1) means, and why there is no reason for why they should equal each other.

**Exercise 7.** Let $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ be numbers, observations, random variables.

(a) Show that

$$\sum_{i=1}^{n} (X_i + Y_i) = \left(\sum_{i=1}^{n} X_i\right) + \left(\sum_{i=1}^{n} Y_i\right).$$

(b) Show that for any numbers $X_1, X_2, \ldots, X_n$ and a number $a$ we have

$$\sum_{i=1}^{n} aX_i = a \sum_{i=1}^{n} X_i.$$

(c) Use (a) and a simple argument to show directly (without expanding the sum) that for any number $a$, we have that

$$\sum_{i=1}^{n} (X_i + a) = \left( \sum_{i=1}^{n} X_i \right) + na.$$

(d) Use (b) and a simple argument to show directly (without expanding the sum) that for any non-zero number $b$, we have that

$$\sum_{i=1}^{n} \frac{X_i}{b} = \frac{\sum_{i=1}^{n} X_i}{b}.$$

**Exercise 8.** Let $X_1, X_2, \ldots, X_n$, $Y_1, Y_2, \ldots, Y_n$ and $a, b, c$ be numbers.

(a) Use Ex. 7(a)&(b) and a simple argument to show directly (without expanding the sum) that we have

$$\sum_{i=1}^{n} (aX_i + bY_i) = a \left( \sum_{i=1}^{n} X_i \right) + b \left( \sum_{i=1}^{n} Y_i \right).$$

(b) Let $A_i = aX_i$ for $i = 1, \ldots, n$. Let $\bar{A}_n = (1/n) \sum_{i=1}^{n} A_i$ be the average of $A_1, A_2, \ldots, A_n$. Show that $\bar{A}_n = a\bar{X}_n$ where $\bar{X}_n = (1/n) \sum_{i=1}^{n} X_i$.

(c) Let $B_i = X_i + Y_i$ for $i = 1, 2, \ldots, n$. Let $\bar{B}_n = (1/n) \sum_{i=1}^{n} B_i$ be the average of $B_1, B_2, \ldots, B_n$. Show that $\bar{B}_n = \bar{X}_n + \bar{Y}_n$.

(d) Let $C_i = aX_i + bY_i + c$ for $i = 1, \ldots, n$. Let $\bar{C}_n = (1/n) \sum_{i=1}^{n} C_i$ be the average of $C_1, C_2, \ldots, C_n$. Show that $\bar{C}_n = a\bar{X}_n + b\bar{Y}_n + c$.

**Exercise 9.** Let $X_1, X_2, \ldots, X_n$ be numbers, observations, random variables.

(a) Define

$$\widetilde{X}_i = X_i - \bar{X}_n, \quad \text{for } i = 1, 2, \ldots, n,$$

where $\bar{X}_n = (1/n) \sum_{i=1}^{n} X_i$. Show that the average of $\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_n$ is zero.

(b) Let us now re-define $\widetilde{X}_i$, and rather set

$$\widetilde{X}_i = \frac{X_i - \bar{X}_n}{s_X}, \quad \text{for } i = 1, 2, \ldots, n,$$

where $s_X$ is the empirical standard deviation of $X_1, X_2, \ldots, X_n$, which is defined to be

$$s_X = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2}.$$

Show that the average of $\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_n$ is equal to zero, and that the standard deviation of $\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_n$ is equal to one.

(c) Let us now re-define $\widetilde{X}_i$, and rather set

$$\widetilde{X}_i = aX_i + b, \quad \text{for } i = 1, 2, \ldots, n,$$

for some numbers $a, b$. Show that

$$\frac{1}{n}\sum_{i=1}^{n} \widetilde{X}_i = a\bar{X}_n + b, \quad \text{and} \quad s^2_{\widetilde{X}} = a^2 s^2_X,$$

where $s^2_{\widetilde{X}}$ is the empirical variance of $\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_n$, and $s^2_X$ is the empirical variance of $X_1, X_2, \ldots, X_n$. Explain why the results of (a) and (b) are special cases of this result.

**Exercise 10.** The empirical variance of $X_1, X_2, \ldots, X_n$ is defined to be

$$s^2_X = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2,$$

and the empirical covariance of $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ is defined to be

$$s_{X,Y} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)(Y_i - \bar{Y}_n),$$

where $\bar{X}_n = (1/n)\sum_{i=1}^n X_i$ and $\bar{Y}_n = (1/n)\sum_{i=1}^n Y_i$. Notice that

$$s_{X,X} = s^2_X.$$

(a) Show that

$$s_{X,Y} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)Y_i = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y}_n)X_i.$$

(b) Show that

$$s^2_X = \left(\frac{1}{n-1}\sum_{i=1}^{n} X_i^2\right) - \frac{n}{n-1}\bar{X}_n^2.$$

**Exercise 11.** When $a < b$, we have

$$\Pr(a < X \leq b) = \Pr(X \leq b) - \Pr(X \leq a). \tag{2}$$

In this exercise we show why this is so. Define the sets

$$A = \{X \leq a\}, \quad \text{and} \quad B = \{a < X \leq b\}.$$

(a) What does $A \cap B$ equal?
(b) What does $A \cup B$ equal?
(c) Use a result from Exercise 3 to prove (2).

## References

Eide, E. (2020). Usikre bevis og juristutdannelsen. *Lov og Rett*, 59:3–18.

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