PROPOSED SOLUTION TO THE FINAL EXAM
GRA6039 AUTUMN 2020

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Exercise 1

(a). Mathematically: $\widehat{\theta}_n$ is a random variable, while θ is a fixed number. Epistemologically: With data, we can realise $\widehat{\theta}_n$, that is, compute a value for it; the value of the parameter θ, on the other hand, will never be fully known.

(b). The estimator $\widehat{\theta}_n$ is unbiased for θ if $\mathrm{E}\,\widehat{\theta}_n = \theta$. (This must be true for every θ, but we give full score to this who mention just the first line here).

(c). That $\widehat{\theta}_n$ is consistent for θ, means that we can pick n such that $\widehat{\theta}_n$ gets arbitrarily close to θ with arbitrarily high probability. You don't need to spell it out like this, it suffices to say that: $\widehat{\theta}_n$ is consistent for θ if for any $\varepsilon > 0$, $\Pr(|\widehat{\theta}_n - \theta| \geq \varepsilon) \to 0$ as $n \to \infty$.

(d). If $\mathrm{E}\,\widehat{\theta}_n = \theta$, and $\mathrm{Var}(\widehat{\theta}_n) \to 0$ as $n \to \infty$, Chebyshev's inequality (Lemma 4.2 in the Lecture notes) gives that for any $\varepsilon > 0$

$$\Pr(|\widehat{\theta}_n - \theta| \geq \varepsilon) \leq \frac{1}{\varepsilon^2}\mathrm{Var}(\widehat{\theta}_n) \to 0, \quad \text{as } n \to \infty,$$

which means that $\widehat{\theta}_n \to_p \theta$.

(e). Use that $F_n(x)$ is 1-to-1,

$$\begin{aligned}\Pr\{F_n^{-1}(1/4) \leq \widehat{\theta}_n \leq F_n^{-1}(3/4)\} &= \Pr\{\widehat{\theta}_n \leq F_n^{-1}(3/4)\} - \Pr\{\widehat{\theta}_n \leq F_n^{-1}(1/4)\} \\ &= \Pr\{\widehat{\theta}_n \leq F_n^{-1}(3/4)\} - \Pr\{\widehat{\theta}_n \leq 1/4\} \\ &= F(F^{-1}(3/4)) - F(F^{-1}(1/4)) = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}.\end{aligned}$$

Exercise 2

(a). To find the expectation and variance, we start by finding $\mathrm{E}\,X^k$, then evaluate for $k = 1, 2$, and use that $\mathrm{Var}(X) = \mathrm{E}\,X^2 - (\mathrm{E}\,X)^2$.

$$\mathrm{E}\,X^k = \int_0^\theta x^k \frac{1}{\theta}\,\mathrm{d}x = \frac{1}{k+1}\frac{1}{\theta}\Big|_0^\theta x^{k+1} = \frac{1}{k+1}\frac{1}{\theta}\theta^{k+1} = \frac{\theta^k}{k+1},$$

from which we get

$$\mathrm{E}\,X = \frac{\theta}{2},$$

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and

$$\mathrm{Var}(X) = \frac{\theta^2}{3} - \frac{\theta^2}{4} = \frac{\theta^2}{12}.$$

(b). The cdf of X is

$$F_X(x) = \int_0^x \frac{1}{\theta}\,\mathrm{d}y = \frac{x}{\theta}, \quad \text{for } 0 \le x < \theta,$$

and $F_X(x) = 0$ for $x < 0$ and $F_X(x) = 1$ for $x \ge \theta$.

(c). Let $Y_n = \max(X_1, \ldots, X_n)$, with $X_1, \ldots, X_n$ independent replicates of X. If $Y_n \le y$, then $X_1 \le y$ and $X_2 \le y$ and $\ldots X_n \le y$, for if one of the X_i were bigger than y, then $Y_n > y$. This means that

$$\{Y_n \le y\} \subset \{X_1 \le y\} \cap \cdots \cap \{X_n \le y\}.$$

Conversely, if $X_i \le y$ for every i, then the biggest of the X_i is clearly smaller than y, so $Y_n = \max(X_1, \ldots, X_n) \le y$, which means that

$$\{Y_n \le y\} \supset \{X_1 \le y\} \cap \cdots \cap \{X_n \le y\}.$$

Inclusion both ways entails that the sets are equal.

Answers along the lines of 'Since Y_n is the biggest of the X_i's, it can only be smaller than y if all the X_i are smaller than y', should gives full score.

(d). The $X_1, \ldots, X_n$ are independent, and $X_i \sim F_X(x)$ for each i. Using what we found in the previous exercise and this independence,

$$\begin{aligned} F_{Y_n}(y) &= \Pr(Y_n \le y) = \Pr(\{X_1 \le y\} \cap \cdots \cap \{X_n \le y\}) \\ &= \Pr(X_1 \le y) \cdots \Pr(X_n \le y) = F_X(y) \cdots F_X(y) = \frac{y}{\theta} \cdots \frac{y}{\theta} = \left(\frac{y}{\theta}\right)^n, \end{aligned}$$

for $y \in [0, \theta)$, and clearly $F_{Y_n}(y) = 0$ for $y < 0$, and $F_{Y_n}(y) = 1$ for $y \ge \theta$.

(e). Start by finding the pdf of Y_n, say $f_{Y_n}(y)$. It is

$$f_{Y_n}(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_{Y_n}(y) = \frac{n}{\theta}\left(\frac{y}{\theta}\right)^{n-1},$$

for $0 \le y \le \theta$, and $f_{Y_n}(y) = 0$ outside this interval. Then $\mathrm{E}\, Y_n^k$ is

$$\begin{aligned} \mathrm{E}\, Y_n^k &= \int_{-\infty}^{\infty} y^k f_{Y_n}(y)\,\mathrm{d}y = \int_0^\theta y^k \frac{n}{\theta}\left(\frac{y}{\theta}\right)^{n-1} \mathrm{d}y = \frac{n}{\theta^n} \int_0^\theta y^{n+k-1}\,\mathrm{d}y \\ &= \frac{n}{\theta^n} \frac{1}{n+k} \bigg|_0^\theta y^{n+k} = \frac{n}{\theta^n} \frac{1}{n+k} \theta^{n+k} = \frac{n\theta^k}{n+k}. \end{aligned}$$

We then find that

$$\mathrm{E}\, Y_n = \frac{n\theta}{n+1},$$

and that

$$\mathrm{Var}(Y_n) = \mathrm{E}\, Y_n^2 - (\mathrm{E}\, Y_n)^2 = \frac{n\theta^2}{n+2} - \frac{n^2\theta^2}{(n+1)^2} = \frac{n\theta^2}{(n+2)(n+1)^2}.$$

Now,

$$\operatorname{E}\widehat{\theta}_1 = \frac{n+1}{n}\operatorname{E}Y_n = \frac{n+1}{n}\frac{n}{n+1}\theta = \theta,$$

which shows that $\widehat{\theta}_1$ is unbiased for θ. The variance of $\widehat{\theta}_1$ is

$$\operatorname{Var}(\widehat{\theta}_1) = \frac{(n+1)^2}{n^2}\operatorname{Var}(Y_n) = \frac{(n+1)^2}{n^2}\frac{n\theta^2}{(n+2)(n+1)^2} = \frac{\theta^2}{n(n+2)}.$$

(f). We have seen that $\operatorname{E}X_i = \theta/2$ for each i. The estimator $\widehat{\theta}_2 = (2/n)\sum_{i=1}^n X_i$ is unbiased,

$$\operatorname{E}\widehat{\theta}_2 = \frac{2}{n}\sum_{i=1}^n \operatorname{E}X_i = \frac{2}{n}\frac{n\theta}{2} = \theta,$$

and, since the X_i's are independent, the variance of this estimator is

$$\operatorname{Var}(\widehat{\theta}_2) = \frac{4}{n^2}\sum_{i=1}^n \operatorname{Var}(X_i) = \frac{4}{n^2}\frac{n\theta^2}{12} = \frac{\theta^2}{3n},$$

(g). Both estimators are unbiased, so we ought to use the estimator with the lower variance.

$$\frac{\operatorname{Var}(\widehat{\theta}_2)}{\operatorname{Var}(\widehat{\theta}_1)} = \frac{\theta^2/(3n)}{\theta^2/\{n(n+2)\}} = \frac{n(n+2)}{3n} = \frac{n+2}{3} \geq \frac{4}{3} > 1,$$

because $n \geq 2$. This shows that $\operatorname{Var}(\widehat{\theta}_1) < \operatorname{Var}(\widehat{\theta}_2)$ for all values of θ, so $\widehat{\theta}_1$ is a better estimator than $\widehat{\theta}_2$.

Exercise 3

We have $S_{t_j} = S_0\exp(\sigma B_{t_j})$ with $B_{t_j} = (1/\sqrt{n})\sum_{i=1}^j Z_i$, for $j = 1,\dots,n$, and $Z_1,\dots,Z_n$ are independent $\mathrm{N}(0,1)$. Then

$$Y_{t_j} = \log S_{t_j} = \log S_0 + \sigma B_{t_j},$$

and

$$Y_{t_j} - Y_{t_{j-1}} = \sigma\frac{1}{\sqrt{n}}Z_j.$$

It is given in the exercise that if $W \sim \mathrm{N}(0,1)$, then $\operatorname{E}W^3 = 0$ and $\operatorname{E}W^4 = 3$. We use this without comment in the following.

(a). The expectation of Z_j^2 is $\operatorname{E}Z_j^2$. Therefore, using linearity of expectation

$$\operatorname{E}\widehat{\sigma}_n^2 = \sum_{j=1}\operatorname{E}(Y_{t_j} - Y_{t_{j-1}})^2 = \sigma^2\frac{1}{n}\sum_{j=1}\operatorname{E}Z_j^2 = \sigma^2,$$

which shows that σ_n^2 is unbiased. Since $\operatorname{E}Z_j^2 = 1$, and

$$\operatorname{Var}(Z_j^2) = \operatorname{E}[Z_j^4] - (\operatorname{E}[Z_j^2])^2 = 3 - 1 = 2,$$

it follows from the Law of large numbers (Theorem 4.3 in the Lecture notes) that

$$\frac{1}{n}\sum_{j=1}Z_j^2 \xrightarrow{p} 1,$$

and we conclude that $\widehat{\sigma}_n^2 \to_p \sigma^2$, that is, $\widehat{\sigma}_n^2$ is *consistent* for σ^2. The really meticulous student might here point to PLIM.2 (Lemma 5.2 in the Lecture notes, or Wooldridge (2019, p. 723)), treating σ^2 is a (constant) sequence, but that is not necessary for a full score on this exercise.

(b). We have seen that $\widehat{\sigma}_n^2 = (1/n)\sum_{j=1}^n \sigma^2 Z_j^2 = \bar{X}_n$, which is the empirical average of $X_1, \ldots, X_n$, where $X_j = \sigma^2 Z_j^2$ for $j = 1, \ldots, n$. Then $X_1, \ldots, X_n$ are independent and identically distributed random variables with expectation σ^2 and variance

$$\mathrm{Var}(X_j) = \mathrm{Var}(\sigma^2 Z_j^2) = \sigma^4 \mathrm{Var}(Z_j^2) = 2\sigma^4.$$

We can therefore use the Central limit theorem (Theorem 5.5 in the Lecture notes, or Wooldridge (2019, p. 724)), to conclude that

$$\frac{\sqrt{n}(\widehat{\sigma}_n^2 - \sigma^2)}{(2\sigma^4)^{1/2}} = \frac{\sqrt{n}(\widehat{\sigma}_n^2 - \sigma^2)}{\sqrt{2}\sigma^2} \overset{d}{\to} \mathrm{N}(0,1),$$

as $n \to \infty$.

(c). The convergence in distribution result above entails that when n is big,

$$\Pr\{\Phi^{-1}(0.025) \leq \frac{\sqrt{n}(\widehat{\sigma}_n^2 - \sigma^2)}{\sqrt{2}\sigma^2} \leq \Phi^{-1}(0.975)\} \approx 0.95.$$

An approximate 95 percent confidence interval is then

$$[\frac{\sqrt{n}\widehat{\sigma}_n^2}{\sqrt{2}\Phi^{-1}(0.975) + \sqrt{n}}, \frac{\sqrt{n}\widehat{\sigma}_n^2}{\sqrt{2}\Phi^{-1}(0.025) + \sqrt{n}}],$$

where $\Phi^{-1}(p)$ is the inverse of the standard normal cdf $\Phi(z)$, so that $\Phi^{-1}(0.975) = -\Phi^{-1}(0.025) = 1.96$. We tacitly assume that n is so that $\sqrt{n} \geq 1.96\sqrt{2} = 2.77$, which is okay, and needs no comment.

(d). Test: Reject $H_0\colon \sigma^2 = 2.34$ vs. $H_A > \sigma^2 = 2.34$, at approximately the 5 perdent significance level, if

$$\frac{\sqrt{n}(\widehat{\sigma}_n^2 - 2.34)}{\sqrt{2}(2.34)} \geq \Phi^{-1}(0.95).$$

The student might verify that the Type I error probability is,

$$\begin{aligned}\Pr(\text{Type I error}) &= \Pr_{2.34}\{\frac{\sqrt{n}(\widehat{\sigma}_n^2 - 2.34)}{\sqrt{2}(2.34)} \geq \Phi^{-1}(0.95)\} \\ &\approx 1 - \Phi(\Phi^{-1}(0.95)) = 1 - 0.95 = 0.05,\end{aligned}$$

but it is not required for a full score.

References

Wooldridge, J. M. (2019). *Introductory Econometrics: A Modern Approach. Seventh Edition.* Cengage Learning, Boston, MA.

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