BI Norwegian Business School

FINAL EXAM: **GRA 6039 – Econometrics with programming**

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HANDED OUT: **05.12.2020 13.00**

DUE DATE: **08.12.2020 13.00**

PERMITTED AIDS: **Lecture notes, books, Google, stackoverflow, etc.**

IMPERMISSIBLE AIDS: **The answer paper must be written and prepared individually. Collaboration with others is not permitted and is considered cheating. All answer papers are automatically subject to plagiarism control. Students may also be called in for an oral consultation as additional verification of an answer paper.**

FORMAT FOR YOUR ANSWER: **A single .pdf-file. You can write on a machine or by hand. None of the questions below require any Matlab programming, and you answer sheet should therefore not include any code.**

INSTRUCTIONS: **Brevity is beautiful. Be as concise as you can. None of the exercises require much text. On Ex. 1(a) your answer should not exceed five lines of text.**

This exam set contains three exercises and comprises two pages.

Exercise 1. Let $\widehat{\theta}_n$ be an estimator of θ, with θ some unknown number on the real line.

(a) **(Max. five lines of text)**. Even though symbolically $\widehat{\theta}_n$ is just θ with a 'hat' on, they are, both mathematically and epistemologically speaking, very different things. In what ways?

(b) Define what it means for $\widehat{\theta}_n$ to be unbiased for θ.

(c) Define what it means that $\widehat{\theta}_n \to_p \theta$ as $n \to \infty$, that is, what it means for $\widehat{\theta}_n$ to be consistent for θ.

(d) Suppose that $\mathrm{E}\,\widehat{\theta}_n = \theta$, and that $\mathrm{Var}(\widehat{\theta}_n) \to 0$ as $n \to \infty$. Show that $\widehat{\theta}_n \to_p \theta$ as $n \to \infty$.

(e) Suppose that $\widehat{\theta}_n$ has cumulative distribution function (cdf) $F_n(x) = \Pr(\widehat{\theta}_n \leq x)$ that is strictly increasing and continuous. Let $F_n^{-1}(p)$ be the inverse function of $F_n(x)$. Compute the probability,

$$\Pr\{F_n^{-1}(1/4) \leq \widehat{\theta}_n \leq F_n^{-1}(3/4)\}.$$

Exercise 2. Let X be a continuous random variable with probability density function (pdf)

$$f_X(x) = \frac{1}{\theta}, \quad \text{for } 0 \leq x \leq \theta,$$

and $f_X(x) = 0$ if x is outside this interval, with $\theta > 0$ some unknown number.

(a) Find expressions for the expectation and the variance of X.

(b) Find the cdf $F_X(x) = \int_{-\infty}^{x} f_X(y)\,\mathrm{d}y$ of X.

(c) Suppose that $X_1, \dots, X_n$ are independent random variables with the same distribution as X, and consider the random variable $Y_n = \max(X_1, \dots, X_n)$, that is, Y_n equals the biggest of the $X_1, \dots, X_n$, with $n \geq 2$. Explain why

$$\{Y_n \leq y\} = \{X_1 \leq y\} \cap \{X_2 \leq y\} \cap \dots \cap \{X_{n-1} \leq y\} \cap \{X_n \leq y\}.$$

Hint: If $A \subset B$ and $B \subset A$, then $A = B$.

(d) Show that the cdf $F_{Y_n}(y)$ of Y_n is
$$F_{Y_n}(y) = \left(\frac{y}{\theta}\right)^n, \quad \text{for } 0 \le y \le \theta,$$
with $F_{Y_n}(y) = 0$ for $y < 0$, and $F_{Y_n}(y) = 1$ for $y \ge \theta$.

(e) Show that
$$\widehat{\theta}_1 = \frac{n+1}{n} Y_n,$$
is an unbiased estimator for θ. Find also the variance of $\widehat{\theta}_1$.

(f) Another estimator for θ is
$$\widehat{\theta}_2 = \frac{2}{n}\sum_{i=1}^{n} X_i.$$
Show that $\widehat{\theta}_2$ is also unbiased for θ, and find an expression for its variance.

(g) Which estimator do you think is best, $\widehat{\theta}_1$ or $\widehat{\theta}_2$? And why? *Hint:* Evaluating the ratio $\mathrm{Var}(\widehat{\theta}_2)/\mathrm{Var}(\widehat{\theta}_1)$ might be easier than evaluating the difference $\mathrm{Var}(\widehat{\theta}_2) - \mathrm{Var}(\widehat{\theta}_1)$.

Exercise 3. The price of an asset is observed over the interval $[0, 1]$, at time points
$$0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1,$$
with $t_j - t_{j-1} = 1/n$ for $j = 1, \ldots, n$. As our model for the price we take
$$S_{t_j} = S_0 \exp(\sigma B_{t_j}), \quad \text{for } j = 1, \ldots, n,$$
with $\sigma > 0$, and $S_0 = 17.89$ the price at time $t_0 = 0$, and
$$B_{t_j} = \frac{1}{\sqrt{n}}\sum_{i=1}^{j} Z_i, \quad \text{for } j = 1, \ldots, n,$$
with $Z_1, \ldots, Z_n$ independent $\mathrm{N}(0, 1)$. In this exercise we want to say something about the volatility σ^2 of the asset. Define
$$Y_{t_j} = \log S_{t_j}, \quad \text{for } j = 0, \ldots, n.$$
As our estimator for σ^2 we use
$$\widehat{\sigma}_n^2 = \sum_{j=1}^{n} (Y_{t_j} - Y_{t_{j-1}})^2.$$
In the following, you might need that if $W \sim \mathrm{N}(0, 1)$, then $\mathrm{E}\,[W^3] = 0$, and $\mathrm{E}\,[W^4] = 3$.

(a) Show that $\widehat{\sigma}_n^2$ is unbiased for σ^2. Show also that it is consistent for σ^2.

(b) Show that
$$\frac{\sqrt{n}(\widehat{\sigma}_n^2 - \sigma^2)}{\sqrt{2}\sigma^2} \xrightarrow{d} \mathrm{N}(0, 1).$$

(c) Find an approximate 95 percent confidence interval for the volatility σ^2.

(d) We would like to test the hypothesis
$$H_0\colon \sigma^2 = 2.34, \quad \text{vs.} \quad H_A\colon \sigma^2 > 2.34.$$
Construct a test for H_0 at the 5 percent significance level.