

## *BI Norwegian Business School*

FINAL EXAM: **EBA 2904 – Statistics with programming**

WITH: **Emil Aas Stoltenberg**

DAY OF EXAMINATION: **June 3, 2025**

EXAMINATION HOURS: **09:00–14:00**

PERMITTED AIDS: **A bilingual dictionary**

INSTRUCTIONS: **Write with a pen, NOT a pencil. Be concise.**

This exam set contains three exercises and comprises four pages. An appendix with results that can be pointed to in solving the exercises is included on pages 3 and 4.

**Exercise 1.** The sample space

$$\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\},$$

is equipped with the probability function

$$\Pr(A) = \frac{|A|}{6}, \quad \text{for all } A \subset \Omega,$$

where  $|A|$  denotes the number of elements in the event  $A$ . You are not asked to show that  $\Pr$  is a probability function. Let  $X$  be a function with domain  $\Omega$  and range  $S = \{s_1, \dots, s_k\}$  for some number  $k \leq 6$ . Define the function

$$P(B) = \Pr(\{\omega \in \Omega: X(\omega) \in B\}), \quad \text{for all } B \subset S.$$

- (a). Show that  $P$  is a probability function on  $S$ .
- (b). Let  $A = \{\omega_1, \omega_3, \omega_5\}$  and define  $Y = I_A$ . Compute the expectation of  $Y$ .
- (c). Let  $B = \{\omega_5, \omega_6\}$  and define  $Z = I_B$ . Show that

$$E(YZ) = E(Y)E(Z).$$

- (d). Define the random variable

$$W(\omega) = 2Y(\omega) - 3Z(\omega), \quad \omega \in \Omega.$$

Find the probability mass function of  $W$ , that is, compute the probabilities  $\Pr(W = w)$  for all real numbers  $w$ .

- (e). Compute the expectation and the variance of  $W$ .
- (f). Make a sketch of the cumulative distribution function  $F(w) = \Pr(W \leq w)$  of  $W$ . Clearly indicate the numbers on the  $x$ -axis and the  $y$ -axis.
- (g). Let  $W_1, \dots, W_n$  be independent and identically distributed random variables, all with the same distribution as  $W$ . Given any  $\varepsilon > 0$  and  $\delta > 0$  we can find  $n_0$  such that

$$\Pr\left(\frac{1}{n} \sum_{i=1}^n W_i \geq \varepsilon\right) < \delta, \quad \text{for all } n \geq n_0.$$

Explain why this is so.

**Exercise 2.** Consider the function

$$f_\theta(x) = 1/\theta, \quad x \in [0, \theta],$$

and  $f_\theta(x) = 0$  when  $x$  is outside of  $[0, \theta]$ , with  $\theta > 0$  an unknown parameter.

(a). Show that  $f_\theta$  is a probability density function.

(b). Let  $X$  be a random variable with probability density function  $f_\theta$ . Find the cumulative distribution function  $F_\theta(x) = \Pr(X \leq x)$  of  $X$ .

(c). For any  $k > 0$  the expectation of  $X^k$  is

$$\mathbb{E} X^k = \frac{\theta^k}{k+1},$$

and you are not asked to show this. But please find the expectation and the variance of  $X$ .

(d). Let  $X_1, \dots, X_n$  be independent and identically distributed random variables all with the same distribution as  $X$ . Define the estimator

$$\hat{\theta}_1 = \frac{2}{n} \sum_{i=1}^n X_i.$$

and find an expression for its mean squared error  $\text{mse}_1(\theta) = \mathbb{E}[(\hat{\theta}_1 - \theta)^2]$ .

(e). Consider the random variable  $M = \max(X_1, \dots, X_n)$ . It can be shown that

$$\Pr(M_n \leq y) = \begin{cases} 0, & y < 0, \\ (y/\theta)^n, & 0 \leq y < \theta, \\ 1, & y \geq \theta. \end{cases}$$

You are not asked to show this. Let  $\hat{\theta}_2 = cM$  be an estimator for  $\theta$ . Find  $c$  such that  $\mathbb{E} \hat{\theta}_2 = \theta$  for all  $\theta > 0$ .

(f). Find an expression for its mean squared error  $\text{mse}_2(\theta) = \mathbb{E}[(\hat{\theta}_2 - \theta)^2]$ . What estimator of  $\theta$  is the winner in terms of the mean squared error?

**Exercise 3.** Fortunately, only one percent of the population is affected by a certain rare disease. There exists two tests to ascertain whether an individual suffers from the rare disease in questions, Test *A* and Test *B*. The producer of Test *A* says that the probability of a positive test (i.e., the test says ‘sick’) given that the individual tested has the disease, is 0.90, while the probability of a false positive is 0.15. The corresponding numbers for Test *B*, the producer of Test *B* informs us, are 0.95 and 0.20.

(a). When choosing among Test *A* and Test *B* for her patient, the doctor wants to use the test that minimises the probability of a wrong test result. Which one is it?

(b). Our doctor finds the previous question difficult, so she decides on taking both tests. Both come out positive. What is the probability that the patient has the disease? What key assumption did you make in making this calculation?

(c). Consider the event {Test *A* positive} and the event {Test *B* positive}. Are these events independent or dependent? Justify your answer.

## APPENDIX

(A.0) The function  $\Pr(\cdot)$  taking events as its arguments, is a probability function if (i)  $\Pr(A) \geq 0$  for all events  $A$ ; (ii)  $\Pr(\Omega) = 1$  for the sample space  $\Omega$ ; and (iii) if  $A$  and  $B$  are disjoint events (i.e.  $A \cap B = \emptyset$ ), then  $\Pr(A \cup B) = \Pr(A) + \Pr(B)$ .

(A.1) For any probability function and events  $A$  and  $B$ , we have

- (a)  $\Pr(\emptyset) = 0$ .
- (b)  $\Pr(A) \leq 1$ .
- (c)  $\Pr(A) = 1 - \Pr(A^c)$ .
- (d)  $\Pr(B \setminus A) = \Pr(B) - \Pr(A \cap B)$ .
- (e)  $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$ .
- (f) If  $A \subset B$  then  $\Pr(A) \leq \Pr(B)$ .

(A.2) The conditional probability of  $A$  given  $B$  is

$$\Pr(A | B) = \frac{\Pr(A \cap B)}{\Pr(B)}, \quad \text{provided } \Pr(B) > 0.$$

This definition gives the multiplication rule  $\Pr(A \cap B) = \Pr(A | B)\Pr(B)$ . A conditional probability function  $B \mapsto \Pr(B | A)$  is a probability function.

(A.3) Let  $A$  and  $B$  be two events, such that  $0 < \Pr(B) < 1$ . The law of total probability says that

$$\Pr(A) = \Pr(A | B)\Pr(B) + \Pr(A | B^c)\Pr(B^c),$$

and if  $B_1, \dots, B_k$  is a partition of the sample space  $\Omega$ , i.e.,  $B_i \cap B_j = \emptyset$  whenever  $i \neq j$  and  $B_1 \cup \dots \cup B_k = \Omega$ , then  $\Pr(A) = \sum_{j=1}^k \Pr(A | B_j)\Pr(B_j)$ .

(A.4) Let  $A$  and  $B$  be two events such that  $\Pr(A) > 0$  and  $\Pr(B) > 0$ . Bayes' formula reads

$$\Pr(B | A) = \frac{\Pr(A | B)\Pr(B)}{\Pr(A)}.$$

(A.5) Two events  $A$  and  $B$  are independent if  $\Pr(A \cap B) = \Pr(A)\Pr(B)$ . The events  $A_1, \dots, A_n$  are mutually independent (or just independent) if

$$\Pr(A_{j_1} \cap \dots \cap A_{j_k}) = \Pr(A_{j_1}) \cdots \Pr(A_{j_k}),$$

for any subset  $\{j_1, \dots, j_k\}$  of  $\{1, \dots, n\}$ .

(A.6) The expectation of a discrete random variable  $W$  taking the values  $\{w_1, w_2, \dots\}$  is

$$\mathbb{E}(W) = \sum_{\omega \in \Omega} W(\omega)\Pr(\omega) = \sum_{j=1}^{\infty} w_j \Pr(W = w_j).$$

The expectation of a continuous random variable  $Z$  with probability density function  $f(z)$  is

$$\mathbb{E}(Z) = \int_{-\infty}^{\infty} z f(z) dz.$$

The variance of any random variable  $Z$  is

$$\text{Var}(Z) = \mathbb{E}\{(Z - \mathbb{E}[Z])^2\} = \mathbb{E}\{Z^2\} - (\mathbb{E}\{Z\})^2.$$

(A.7) The indicator function of an event  $A$ , denoted  $I_A$ , is

$$I_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A, \\ 0, & \text{otherwise.} \end{cases}$$

The expectation of an indicator function is  $E(I_A) = \Pr(A)$ . For any two events  $A$  and  $B$ , the product of their indicator functions is  $I_A I_B = I_{A \cap B}$ .

(A.8) The cumulative distribution function of a random variable  $X$  is the function  $F(x) = \Pr(X \leq x)$ . If  $X$  is a discrete random variable taking the values  $\{x_1, x_2, \dots\}$ , then

$$F(x) = \sum_{j: x_j \leq x} \Pr(X = x_j).$$

If  $X$  is a continuous random variable with p.d.f.  $f(x)$ , then

$$F(x) = \int_{-\infty}^x f(y) \, dy.$$

In particular, the p.d.f. of a continuous random variable can be found by differentiating its c.d.f., i.e.,

$$f(x) = F'(x).$$

(A.9) If  $\hat{\theta}_n$  is an estimator of  $\theta$ , the mean squared error of  $\hat{\theta}_n$  is

$$\text{mse} = E\{(\hat{\theta}_n - \theta)^2\}.$$

It is always true that  $\text{mse} = \text{Var}(\hat{\theta}_n) + \{E(\hat{\theta}_n) - \theta\}^2$ . Notice that the mse is a function of  $\theta$ . Small mse is good.

(A.10) Markov's inequality: Let  $X$  be a nonnegative random variable. For any  $\varepsilon > 0$ , we have that  $\Pr(X \geq \varepsilon) \leq E(X)/\varepsilon$ . We deduce that for any random variable  $Y$

$$\Pr(|Y| \geq \varepsilon) \leq \frac{1}{\varepsilon^2} E(Y^2).$$

for any any  $\varepsilon > 0$ .